Math 635 – Spring 2010

Homework 6

Due: April 29, 2010, beginning of the class. Late homework will **not** be accepted.

1. (a) Suppose that $X(t) = (X_1(t), X_2(t)) \in \mathbb{R}^2$ satisfies

$$\begin{aligned} dX_1 &= -X_2 dB \\ dX_2 &= X_1 dB \end{aligned}$$

and $|X(0)| = 1$. Show that $X(t)$ does *not* stay on the circle $|X| = 1$, even though the vector field $(-X_2, X_1)$ is tangent to the circle at (X_1, X_2).
(Hint: Apply Itô to $|X(t)|^2$.)

(b) Show that if $X(t) = (X_1(t), X_2(t))$ satisfies

$$\begin{aligned} dX_1 &= -X_2 dB - \frac{1}{2} X_1 dt \\ dX_2 &= X_1 dB - \frac{1}{2} X_2 dt \end{aligned}$$

with $|X(0)| = 1$ then $X(t)$ stays on the unit circle.

2. Show that if B_t is a standard BM then $M_t = (B_t + t)\exp(-B_t - t/2)$ is a martingale.

3. **(Exercise 9.2)** Solve the SDE

$$dX_t = tX_t dt + e^{t^2/2} dB_t, \qquad X_0 = 1.$$

4. **(Exercise 9.3)** Solve the SDE

$$X_0 = 0, \quad dX_t = -2\frac{X_t}{1-t} dt + \sqrt{t(1-t)} dB_t, \quad 0 \le t \le 1.$$

Show that the solution is a Gaussian process and find the covariance function $Cov(X_s, X_t)$.

Bonus problem.
(Exercise 8.5) Let $\underline{B}(t)$ be a 3-dimensional standard BM. Let $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$, show that $M_t = f(\underline{B}_t), 1 \le t < \infty$ is an L^2 bounded local martingale, but not a martingale.