Math 635 – Spring 2010

Homework 5

Due: April 8, 2010, beginning of the class. Late homework will **not** be accepted.

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, prove (without the use of Itô's formula) that

$$\int_0^T f(s)dB_s = f(T)B_T - \int_0^T f'(s)B_s ds.$$

2. (a) Let B_t be a standard Brownian motion, f_1 and f_2 functions on $[0, 1]$ and let $X_i = \int_0^1 f_i(t)B_t dt$. Find the joint distribution of $X = (X_1, X_2)$.
   Hint: It will be a multidimensional normal.

   (b) Compute the Fourier series of Brownian motion on $[0, 1]$, i.e. compute all finite dimensional distributions of the sequence $a_n = \int_0^1 \exp(i2\pi nt)B_t dt$.

3. **(Exercise 7.1)** Find a function τ_t such that the processes

$$X_t = \int_0^t e^s dB_s, \qquad Y_t = B_{\tau_t}$$

   have the same distributions. Use this to compute EX_t^4 and $P(X_t \geq 1)$.

4. **(Exercise 7.2)** Show that if X_t is a continuous local submartingale with

$$E(\sup_{0 \leq s \leq T} |X_s|) < \infty$$

   then $X_t, 0 \leq t \leq T$ is an honest submartingale.
   Hint: Try to follow the argument in Proposition 7.10.