Math 635 – Spring 2010

Homework 4

Due: March 16, 2010, beginning of the class. Late homework will **not** be accepted.

1. **(Exercise 6.1)** Use the Itô isometry to calculate the variances of

$$\int_0^t |B_s|^{1/2} dBs \quad \text{and} \quad \int_0^t (B_s + s)^2 dB_s$$

2. **(Exercise 6.2)** The integrals

$$I_1 = \int_0^t B_s ds \quad \text{and} \quad I_2 = \int_0^t B_s^2 ds$$

are not stochastic integrals although they are random variables. In these cases we just integrate a (random) continuous function the usual (traditional) way. Find the mean and variance of I_1 and I_2.
Also: compute the moment generating function of I_1. (It's easy once you have the mean and variance...)

3. Prove the following (weaker) version of the iterated logarithm theorem:

$$\limsup_{t\to\infty} \frac{B_t^*}{\sqrt{2t \log\log t}} \le 1 \quad \text{a.s.}$$

where $B_t^* = \max_{0\le s\le t} B_s$.
Hint: Let $t_n = c^n$ with $c > 1$ and show that $B_{t_n}^* \ge c\sqrt{2t_n \log\log t_n}$ a.s. if n large enough.

4. Use the definition of the Itô integral to prove that

$$\int_0^t B_s^2 dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s ds.$$

Bonus problem.

(a) Let X_t be a continuous non-negative martingale with $X_0 = 1$ and $X_t \to 0$ a.s. Show that for any $x \ge 1$

$$P(\sup_{t\ge 0} X_t \ge x) = 1/x.$$

Hint: consider the hitting time τ_x of x and the stopped martingale $X_{t\wedge\tau_x}$.

(b) Use the previous result to show that if $b > 0$ then $Y = \sup_t(B_t - bt)$ is an exponential random variable with parameter $2b$. (I.e. $P(Y \le x) = 1 - e^{-2bx}$ if $x \ge 0$.)