Math 635 – Spring 2010

Homework 3

Due: March 2, 2010, beginning of the class. Late homework will **not** be accepted.

1. **(Exercise 4.1)**

   (a) (Tower property) Use the definition of conditional expectation to prove that if $\mathcal{H}$ is a sub-σ field of $\mathcal{G}$ then
   $$E(E(X|\mathcal{G})|\mathcal{H}) = E(X|\mathcal{H}).$$

   (b) If $Y \in \mathcal{G} \subset \mathcal{F}$ (where $\mathcal{F}$ is the σ-field of the probability space), $E|X| < \infty$ and Y is bounded then
   $$E(XY|\mathcal{G}) = YE(X|\mathcal{G}).$$
   Hint: we can approximate Y with simple rv's of the form $\sum c_i 1_{A_i}$ with $A_i \in \mathcal{G}$, so it is enough to prove the identity for such rv's.

2. **(Exercise 4.2 (c))** If $X_n \to X$ in probability and $X_n \to Y$ a.s. then $P(X = Y) = 1$.

3. **(Exercise 4.4 (c))** Prove that $t^{-1}B_t$ converges to 0 almost surely as $t \to \infty$.
   Hint: one possibility is to use the maximal inequality to estimate
   $$P(\sup\{|B_t| : 2^n \le t \le 2^{n+1}\} > 2^n)$$
   and then use the Borel-Cantelli lemma. Another way would be to use the strong law of large numbers for $X_n = B_n - B_{n-1}$ and show that $t^{-1}B_t$ and $n^{-1}B_n$ are close if n is close to t.

4. Show that with probability one the standard BM is not strictly increasing on any interval.
   Hints: First show that it is enough to prove that $P(B_t$ is strictly increasing on $[a,b]) = 0$ for any $0 \le a < b$. Then use the fact that
   $$P(B_t \text{ is strictly increasing on } [a,b]) \le P(B_{t_1} < B_{t_2} < \dots B_{t_k}), \quad \text{if} \quad a \le t_1 < t_2 < \dots \le b.$$

Bonus problem. Find the probability that B_t eventually hits the line $a + bt$ (where $a, b > 0$).