Math 635 – Spring 2010

Homework 2

Due: February 16, 2010, beginning of the class. Late homework will **not** be accepted.

1. **(Exercise 2.2)** Assume that a sequence of *independent* events $\{A_i\}$ satisfy $\sum_{n=1}^{\infty} P(A_i) = \infty$. Let
$$\tau_k = \min\{n : \sum_{i=1}^{n} 1_{A_i} = k\}.$$
By the (second) Borel-Cantelli lemma with probability one we will have infinitely many of the A_i's occurring, i.e. $\sum_{k=1}^{\infty} 1_{A_k} = \infty$ and $P(\tau_k < \infty) = 1$ for all k. Prove the slightly stronger statement
$$k = E \sum_{i=1}^{\tau_k} P(A_i).$$
Why is this a stronger statement?

   Hint: construct a martingale using the random variables 1_{A_i} and use the fact that τ_k is a stopping time.

2. In each of the following cases check if the process is a standard Brownian motion.
   (a) $X_t = \frac{1}{\sqrt{t}} B_{t^2}$ where B_t is a standard BM.
   (b) $Y_t = \sin(\alpha) B_t^{(1)} + \cos(\alpha) B_t^{(2)}$ where $B_t^{(1)}$ and $B_t^{(2)}$ are independent standard BM's and $\alpha \in \mathbf{R}$.
   (c)
$$Z_t = \begin{cases} B_t & 0 \le t \le 1 \\ B_{t+1} - B_2 + B_1 & t \ge 1 \end{cases}$$
   where B_t is a standard BM.

3. **(Exercise 3.1 (b)-(d))** Let U_t be a standard Brownian bridge (see page 41 for the definition).
   (a) Show that $Cov(U_s, U_t) = s(1-t)$ for $0 \le s \le t \le 1$.
   (b) Let $X_t = g(t) B_{h(t)}$, and find functions g and h such that X_t has the same covariance as the Brownian bridge. (B_t is a standard BM.)
   (c) Show that $Y_t = (1+t) U_{t/(1+t)}$ is a BM on $[0, \infty)$.

   Hint: (c) should help with (b)...

4. An urn contains a red and b black balls. In each step we draw a ball randomly and replace it with two balls of the same color. (Essentially in each step we add a new ball to the urn whose color is determined randomly.) Let X_n be the ratio of red balls in the urn after the n^{th} step. ($X_0 = a/(a+b)$.)

   Show that X_n is a martingale and it converges almost surely.

Bonus problem. Assume that X and Y are independent, identically distributed with mean 0 and variance 1. Show that if the random variables $X+Y, X-Y$ are independent then X and Y are standard normals.