

# Math 635 – Spring 2010

## Take-home Final Exam

SOLVE FOUR OUT OF THE FIVE PROBLEMS!

**Due: May 13, 2010, 10 AM.**

You may hand in your exam either to the lecturer's mailbox or via email.

- Late exams will **not** be accepted.
- You are allowed to use anything, but it would be great if you could solve the problems using only the textbook and your notes.
- Try to give a proper reference whenever you want to use something that we proved in class.
- Your solutions should be concise, clear and legibly written.
- You may not collaborate with other students.

1. Let  $a, b > 0$ ,  $\theta \in \mathbb{R}$  and consider the process

$$X_t = f(t) \cosh \left( \theta \left( B_t - \frac{a-b}{2} \right) \right)$$

where  $B_t$  is a standard BM and  $f(t)$  is a deterministic function.

- (a) Find an  $f(t)$  for which  $X_t$  is a local martingale. Show that it is actually an honest martingale.  
(Recall that  $\cosh x = (e^x + e^{-x})/2$ .)
- (b) Use part (a) to compute the moment generating function of the hitting time  $\tau = \inf\{t : B_t = a \text{ or } B_t = -b\}$ . Prove that

$$\mathbb{E} [e^{-\lambda\tau}] = \frac{\cosh(\sqrt{2\lambda}(b-a)/2)}{\cosh(\sqrt{2\lambda}(b+a)/2)}, \quad \text{for } 0 \leq \lambda.$$

(Don't forget to prove every step in your argument!)

2. Assume that  $X_t$  satisfies the SDE  $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$ . Find a similar SDE for  $Y_t = X_{t^2}$  (possibly with a different BM).

Hint: If  $\sigma = 0$  then this is just a simple change of variables. You might want to look at the case when  $X_t = B_t$ , Corollary 7.1 in the book will help. With these you should be able to take care of the general case.

3. Solve the SDE

$$dX_t = -2(X_t - 1)dt + 3dB_t, \quad X_0 = 2.$$

Compute  $\text{Var}X_t$  for your solution.

Hint: this is very similar to one of the examples that we discussed in class. Try to modify the discussed solution to solve this equation or transform this one into the discussed form. If you do not succeed, try to compute  $\text{E}X_t$  and compare it to the formula in the discussed example, this should give you a hint about the solution.

4. Functions of the form  $f : \mathbb{C} \rightarrow \mathbb{C}$  can be represented as  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  functions if we write  $f(x+iy) = u(x, y) + i \cdot v(x, y)$ . (I.e. if we look at the complex plane as  $\mathbb{R}^2$ .) The function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is said to be differentiable in the complex sense at  $z_0$  if  $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$  exists in  $\mathbb{C}$ . It is known that if  $f$  is differentiable everywhere in  $\mathbb{C}$  then

$$u_x(x, y) = v_y(x, y), \quad u_y(x, y) = -v_x(x, y)$$

for all  $x, y \in \mathbb{R}$  and  $u, v$  are smooth functions.

(a) Use this to derive the complex version of the Itô formula:

If  $f : \mathbb{C} \rightarrow \mathbb{C}$  is differentiable on  $\mathbb{C}$  and  $Z(t) = B_1(t) + iB_2(t)$  is a complex Brownian motion with  $B_1, B_2$  independent standard Brownian motions then

$$df(Z) = f'(Z)dZ \quad \text{where } dZ = dB_1 + idB_2.$$

Note that there is no extra  $dt$  term!

Hint: rewrite everything with vector valued functions and use the appropriate version of the Itô formula. If  $f(x + iy) = u(x, y) + i \cdot v(x, y)$  then  $f'(x + iy) = u_x(x, y) + i \cdot v_x(x, y)$ .

(b) Give an easy proof of the fact that if  $X, Y$  are i.i.d. standard normals then for any positive integer  $n$  we have  $\text{E}(X + iY)^n = 0$ .

(You can use that  $z \rightarrow z^n$  is differentiable everywhere in  $\mathbb{C}$ .)

5. Assume that the process  $X_t$  satisfies the SDE  $dX_t = e^{-t}dt + X_tdB_t$  with  $X_0 = 1$ .

(a) Find a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  so that  $Y_t = g(X_t)$  satisfies an SDE of the form  $dY_t = f(t, Y_t)dt + dB_t$  with some function  $f$ .

(b) Compare your result from part (a) to a Brownian motion with a drift and conclude that  $X_t$  stays positive for all  $t \geq 0$  with probability 1.

(c) **(Bonus)** Show that  $X_t$  converges to 0 with probability 1.