MATH 275 – Review sheet for the final exam

Time: December 18, Tuesday, 12:25PM - 2:25PM
Place: Ingraham 120

This is an outline to help you study for the final exam. It is meant to give you a sample of problems, concepts, topics you may encounter on the exam. Note that anything that we covered in class is fair game: you might see problems or questions which are not explicitly listed here. You should definitely know how to do the homework problems that have been assigned. In general, you don't have to reproduce the *long* proofs that we did in class, but you should understand them. (This might help you with the solution of some of the problems.)

We have covered the following topics since the last midterm:

Differentiation

You should know

- the definition of the derivative
- compute the derivative using the limit definition
- the various differentiation rules (sum, product, ratio and chain rule). You should be able to prove the simpler ones.
- how to solve related rate problems
- how to find the equation for the tangent line for a differentiable function
- the method of implicit differentiation
- how to find the derivative of the inverse function
- Rolle's theorem and the Mean Value Theorem for derivatives
- how to find and identify relative extrema
- how to find the absolute max/min on an interval or on $\mathbb{R}$
- how to solve word problems related to extreme value problems
- how to decide if a function is monotone increasing/decreasing /convex/concave on an interval
- how to sketch the graph of a given function (using the steps discussed in class)

Connection between integration and differentiation

You should know

- the two Fundamental Theorems of Calculus
- the definition of the primitive function
- the Zero Derivative theorem
- how to use substitution and integration by parts to compute integrals

The logarithm and exponential function

You should know

- the integral definition of the logarithm function and its various properties
- the exponential function as the inverse of the logarithm function and its various properties
- (If we have time to cover it in class): the inverse trigonometric functions

Sample practice problems

1. Differentiate the following functions:
   (a) $\cos(x^2 + \sin x)$
   (b) $e^{x^2} x^3 \sqrt{x}$
   (c) $\log(\log(\log x))$
2. Find the absolute maximum and absolute minimum of these functions:
   (a) $\frac{x}{x^2+25}$ on $[-6, 4]$
   (b) $\frac{x}{x^2+25}$ on $\mathbb{R}$
   (c) $e^{-x^2} x$ on $\mathbb{R}$
   (d) $x \log x$ on $[1/2, 2]$
3. Sketch the functions in the previous problem.
4. Various optimization problems from the book: **4.21: 3, 7, 10 13, 16, 23**
5. Show that if $f'(x) \leq g'(x)$ for all $x \in [a, b]$ and $f(a) = g(a)$ then $f(x) \leq g(x)$ for all $x \in [a, b]$. (Hint: consider $f - g$.)
6. Show that for all $x > 0$ we have $\sqrt{1 + x} < 1 + x/2$. (Hint: use the previous problem.)
7. Show that if $f(x)$ is twice differentiable on $\mathbb{R}$ and it is concave everywhere then the tangent line at $x = x_0$ will be above the graph of the function f for all $x \neq x_0$. (Hint: compare the derivatives.)
8. Show that for any $x > 1$ we have $1 - 1/x < \log x < x - 1$.

9. Find the primitive functions of $f(x) = \sqrt{|x|}$ on $\mathbb{R}$. (You can express the result in terms $|x|$ or as a case defined function.)

10. Practice integration problems:

    (a) $\int \frac{x^3+x+1}{(x^4+2x^2+4x+5)^{3/4}} dx$

    (b) $\int_0^{\pi/2} \cos(t) \sin^7(t) dt$

    (c) $\int x^2 \cos(x^3+1) dx$

    (d) $\int \sin^7(2x+1) dx$

    (e) $\int (x+2)^2 \sqrt[3]{x+1} dx$

    (f) $\int \frac{\cos x}{(\sin x+2)^3} dx$

    (g) $\int_{-\pi}^{\pi} x \sin x dx$

    (h) $\int_0^2 \sqrt{16-x^2} dx$ (Hint: try substitution with a suitable trigonometric function.)

11. Find $G'(x)$ if the function $G(x)$ is defined as

$$G(x) = \int_{\sin x}^{x^{3/2}} \frac{\sqrt{y^2+2}}{y^4+3-\cos y} dy.$$

12. Assume that for a twice differentiable function on $[0,1]$ we have $f(0) = f'(0) = 0$ and $|f''(x)| \leq 1$ for all $x \in (0,1)$. Show that $|f(x)| \leq x^2/2$ in $[0,1]$. (Hint: try to bound $f'(x)$ first.)

13. Show that the function $A(x) = \int_1^x \frac{\sin y}{y^{3/2}} dy$ is bounded on $[1,\infty)$. (Don't try to compute the integral, just estimate it.)

14. Show that function $A(x) = \int_1^x \frac{1}{y^{1/2}} dy$ is not bounded on $[1,\infty)$, but $B(x) = \int_1^x \frac{\sin y}{y^{1/2}} dy$ will be bounded. (Hint: use partial integration and the previous problem.)

15. Find all differentiable functions $f(x)$ for which we have $f(0) = 1$ and $f'(x) = -xf(x)$ (for all x).
    (Hint: move all the $f(x)$-dependent terms to one side, and then integrate both sides of the equation with respect to x.)

Ask for help if you think you need it

If you are having trouble with certain type of problems or concepts then you should ask for help. Come to one of my or Jo's office hours and ask questions! If you think you may have trouble solving problems with a time limit then collect a couple (say six-seven) problems similar to homework problems and try solving them in 120 minutes (with the solutions written up neatly). Remember that it is almost as important that you can present your solutions clearly as it is to actually find those solutions.

GOOD LUCK!