MATH 275 – Review sheet for the second midterm exam

Time: Wednesday, November 14, 7:15PM-8:45PM,
Place: Social Sciences 6102 (NOTE THAT THIS IS NOT OUR USUAL CLASSROOM!)

This is an outline to help you study for the midterm exam. It is meant to give you a sample of problems, concepts, topics you may encounter on the exam. Note that anything that we covered in class is fair game: you might see problems or questions which are not explicitly listed here. You should definitely know how to do the homework problems that have been assigned. In general, you don't have to reproduce the *long* proofs that we did in class, but you should understand them. (This might help you with the solution of some of the problems.)

We have covered the following topics since the last midterm:

Integration

You should know

- the definitions of the upper and lower integral of a function and integrability
- the various properties of integrals
- the theorems about the integrability of certain special classes of functions (monotonic functions, continuous functions)
- how to integrate x^p and $x^{1/p}$ for p positive integer, how to integrate $\sin x, \cos x$
- how we can use the integration to compute areas of various regions (bounded by graphs of functions in the usual and in the polar coordinate system)
- how to compute volumes of certain solids
- how to compute integral averages
- the statement of the Mean Value Theorem and how to use it to estimate integrals

Limits and continuity

You should know

- the definition of the limit (and one-sided limit) using the neighborhoods and also $\varepsilon - \delta$.
- the definition of infinite limits and limits at infinity
- how to apply the definition to prove the existence of the limit (i.e how to do $\varepsilon - \delta$ proofs)

- all the basic limit laws
- how to apply the limit laws to compute limits
- the Squeezing Principle
- the definition of continuity at a given point
- how to apply the basic limit laws to show the continuity of certain functions (e.g. polynomials, rational functions in their domain, etc.)
- the theorem about the composition of continuous functions
- the limit of $\frac{\sin x}{x}$ as $x \to 0$ and how to use this to compute other limits
- Bolzano's theorem and the Intermediate Value Theorem and how to use them to show that certain equations have solutions
- the Extreme Value Theorem and how to apply it
- the definition of uniform continuity

Sample practice problems

1. We know that g is integrable on $[2,7]$ and $\int_2^7 g(x)dx = 5$. Find $\int_0^1 g(2x+5)dx$.
2. Is it true that if $f+g$ is integrable on $[a,b]$ then both f and g must be integrable there?
3. Show that the following function is integrable on $[0,2]$:

$$f(x) = \begin{cases} x & \text{if} \quad x \neq 1 \\ 4 & \text{if} \quad x = 1 \end{cases}$$

   Hint: this is almost continuous, except at $x = 1$.
4. More general version: show that if f is an integrable function on $[a,b]$ and g is equal to f except for maybe finitely many points in $[a,b]$ then g is also integrable.

   Hint: take the difference.
5. Compute the following integrals:
   (a) $\int_1^4 (x^4 - 3x^2 + 2x - 1)dx$
   (b) $\int_{-5}^3 |x^2 - 4|dx$
   (c) $\int_2^7 (\sqrt{x+2} - x^{1/3})dx$
   (d) $\int_{\pi/2}^{2\pi/3} \cos(2x + \pi/6)dx$
   (e) $\int_0^{13\pi/6} |\sin x|dx$

6. Suppose that f is strictly increasing and continuous on $[0, \infty)$ with range $[0, \infty)$ (which also means $f(0) = 0$). Let $g(x)$ be the inverse function of f and $F(x) = \int_0^x f(y)dy$. Find an expression for $\int_0^x g(y)dy$ in terms of F, g.
Hint: copy the proof of the integral of $x^{1/p}$.

7. Find $\lim_{x\to 0} \frac{\sin(2x)+\cos(x^3)x^2}{x}$.

8. Find $\lim_{x\to\infty} \sqrt{x}(\sqrt{x+2} - \sqrt{x})$.

9. Show that if $\lim_{x\to a} f(x) = 0$ then $\lim_{x\to a} \frac{1}{|f(x)|} = \infty$.

10. Show that if $\lim_{x\to\infty} f(x) = 0$ then $\lim_{x\to\infty} \frac{1}{f(x)}$ might not exists (not even in the infinite sense).

11. Is there a function f which is continuous on $\mathbb{R}$ and $f(1/n) = (-1)^n$ for all positive integer n?

12. Is there a function f which is continuous on $\mathbb{R}$ and $f(1/n) = (-1)^n/n$ for all positive integer n?

13. We define the positive part of a number x as

$$x_+ = \begin{cases} x & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

    (a) Show that the function $x \to x_+$ is continuous everywhere.
    (b) Assume that g is continuous everywhere. Show that $g_+(x)$ (the positive part of $g(x)$) is also continuous everywhere.
    (c) Express the maximum of x and y using the positive part function. Use that to show that if f and g are continuous functions then the function $h(x) = \max(f(x), g(x))$ is also continuous.

14. Show that the equation $x^3 = \sin(x^2) + \cos(4x + 3)$ has at least one solution.

15. Show that the function $x - \tan x$ has infinitely many zeros.

16. The function f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 12$. Show that f is bounded on $[0, \infty)$.

Ask for help if you think you need it

If you are having trouble with certain type of problems or concepts then you should ask for help. Come to one of my or Jo's office hours and ask questions! If you think you may have trouble solving problems with a time limit then collect a couple (say five) problems similar to homework problems and try solving them in 90 minutes (with the solutions written up neatly). Remember that it is almost as important that you can present your solutions clearly as it is to actually find those solutions.

GOOD LUCK!