

# Infinite limits, limits at infinity

Math 275 – Fall 2012

As we discussed in class, we can define infinite limits and limits at infinity just by defining the neighborhoods of  $\infty$  and  $-\infty$ . The neighborhoods of  $\infty$  are the intervals of the form  $(c, \infty)$  and the neighborhoods of  $-\infty$  are the intervals of the form  $(-\infty, c)$ . Now we can take the definition of the limit:

We say that  $\lim_{x \rightarrow p} f(x) = A$  if for every neighborhood  $N_1(A)$  of  $A$  there is a neighborhood  $N_2(p)$  of  $p$  so that if  $x \in N_2(p), x \neq p$  then  $f(x) \in N_1(A)$ .

and we may choose  $p$  or  $A$  (or both) to be  $\infty$  or  $-\infty$ . (Note that if  $p$  is  $\pm\infty$  then we don't have to worry about the  $x \neq p$  condition.) The same definitions hold for one-sided limits.

The definitions can be rewritten with the ' $\varepsilon - \delta$ ' language, with inequalities. E.g.

- $\lim_{x \rightarrow 5} f(x) = \infty$  if for any  $c$  there is a  $\delta > 0$  so that if  $0 < |x - 5| < \delta$  then  $f(x) > c$ .
- $\lim_{x \rightarrow -\infty} f(x) = 3$  if for any  $\varepsilon > 0$  there is a  $c$  so that if  $x < c$  then  $|f(x) - 3| < \varepsilon$ .

**Example 1.** Show that  $\lim_{x \rightarrow 3^-} \frac{1}{x-3} = -\infty$

*Solution.* Fix  $c$ , since the limit is  $-\infty$  we may assume that  $c < 0$ . We need a  $\delta > 0$  so that if  $3 - \delta < x < 3$  then  $\frac{1}{x-3} < c$ . But we can actually solve this inequality, assuming  $x < 3$  (which we can) we get  $3 + 1/c < x$ . (Be careful when you are solving this inequality: both  $x - 3$  and  $c$  are negative.) This means that we can choose  $\delta = -\frac{1}{c}$ .  $\square$

**Example 2.** Show that  $\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$ .

*Solution.* Fix  $\varepsilon > 0$ . We need  $c$  so that if  $x > c$  then  $1/x^2 < \varepsilon$ . Choosing  $c = 1/\sqrt{\varepsilon}$  works.  $\square$

## Basic limit laws

One can show that the basic limit laws hold for limits at  $\pm\infty$ . So if  $\lim_{x \rightarrow \infty} f(x) = A$ ,  $\lim_{x \rightarrow \infty} g(x) = B$  where  $A, B$  are real numbers (i.e. not infinite!) then  $f(x) + g(x), f(x) - g(x), f(x)g(x)$  will also have limits at  $\infty$  ( $A+B, A-B$  and  $AB$ ), and if  $B \neq 0$  then  $f(x)/g(x)$  will have a limit  $A/B$ . (Similar statements hold at  $-\infty$ .) You can also use the Squeezing Principle, this is especially useful if you can estimate complicated functions with simpler ones.

If you want to use the limit laws for infinite limits then you have to be a bit more careful: remember that  $\infty$  and  $-\infty$  are not real numbers. However you can still 'pretend' that certain operations can be carried out:

' $\infty + \infty = \infty$ ', ' $\infty \times \infty = \infty$ ', ' $c \times \infty = \infty$ ' if  $c > 0$  and ' $c \times \infty = -\infty$ ' if  $c < 0$ , ' $\frac{c}{\infty} = 0$ ', ' $c + \infty = \infty$ ' etc. You should actually use these as rules for limits, not as the outcomes of operations on  $\infty$ . That means that you will have statements like this:

Assume that  $\lim_{x \rightarrow p} f(x) = \infty$  and  $\lim_{x \rightarrow p} g(x) = \infty$ . Then

- $\lim_{x \rightarrow p} (f(x) + g(x)) = \infty$

- $\lim_{x \rightarrow p} f(x)g(x) = \infty$
- $\lim_{x \rightarrow p} \frac{4}{f(x)} = 0$ .

Note that if  $f$  and  $g$  have  $\infty$  limits at  $p$  then we cannot say anything about the limit of  $f(x) - g(x)$  and  $f(x)/g(x)$  there. (It can be basically anything or it might not exist.)

## Polynomials and rational functions

It is not hard to show that for any non-constant polynomial  $p(x)$  the limit of  $p(x)$  will be  $\infty$  or  $-\infty$  as  $x \rightarrow \infty$  or  $x \rightarrow -\infty$ . (Eventually the main term of the polynomial will dominate, so you will need to find the limit of  $cx^n$  as  $x \rightarrow \pm\infty$ .)

For rational functions the limits at infinity can also be a real number, this will happen if the degree for the numerator is at most as big as the degree of the denominator.

**Example 3.** Show that  $\lim_{x \rightarrow \infty} \frac{x^2 - 3x + 4}{2x^2 + 4x - 5} = \frac{1}{2}$ .

*Solution.* In order to find the limit at  $\infty$ , it is enough to evaluate the function for large values of  $x$ . If  $x > 0$  then  $\frac{x^2 - 3x + 4}{2x^2 + 4x - 5} = \frac{1 - \frac{3}{x} + \frac{4}{x^2}}{2 + \frac{4}{x} - \frac{5}{x^2}}$  whenever both sides make sense. But now the numerator has a limit equal to 1 as  $x \rightarrow \infty$  and the limit of the denominator as  $x \rightarrow \infty$  is 2 so the limit of the ratio is  $\frac{1}{2}$ .  $\square$

One can also show that if the rational function  $f(x) = \frac{p(x)}{q(x)}$  is not defined at  $x = c$  (because  $q(c) = 0$ ) then  $\lim_{x \rightarrow c^-} f(x)$  and  $\lim_{x \rightarrow c^+} f(x)$  will always exist (but might be infinite).

**Example 4.** Show that  $\lim_{x \rightarrow 2^-} \frac{x^2 + 2}{x - 2} = -\infty$  and  $\lim_{x \rightarrow 2^+} \frac{x^2 + 2}{x - 2} = \infty$ .

*Solution.*  $\lim_{x \rightarrow 2} x^2 + 2 = 4$  and the one-sided limits of  $\frac{1}{x - 2}$  are  $\infty$  from the left and  $\infty$  from the right. From this the statement follows.  $\square$

## Practice problems

1. Write out the definition of the following statements using inequalities:

- (a)  $\lim_{x \rightarrow 2^+} g(x) = \infty$
- (b)  $\lim_{x \rightarrow \infty} h(x) = -\infty$

2. Find the following limits (you may use the limit laws)

- (a)  $\lim_{x \rightarrow 2} \frac{1}{(x - 2)^2}$
- (b)  $\lim_{x \rightarrow 4^-} \frac{x}{4 - x}$
- (c)  $\lim_{x \rightarrow -\infty} \frac{1}{x^3}$

3. Assume that  $\lim_{x \rightarrow 2^-} f(x) = \infty$  and  $\lim_{x \rightarrow 2^-} g(x) = 5$ . Show (with ' $\varepsilon - \delta$ ' proofs) that

- (a)  $\lim_{x \rightarrow 2^-} f(x)^2 = \infty$
- (b)  $\lim_{x \rightarrow 2^-} f(x)g(x) = \infty$
- (c)  $\lim_{x \rightarrow 2^-} g(x) - f(x) = \infty$

- (d)  $\lim_{x \rightarrow 2^-} g(x)/f(x) = 0$ .
4. Give examples of functions  $f(x)$ ,  $g(x)$  for which we have  $\lim_{x \rightarrow 0} f(x) = \infty$ ,  $\lim_{x \rightarrow 0} g(x) = \infty$  and
- (a)  $\lim_{x \rightarrow 0} f(x) - g(x) = 5$
  - (b)  $\lim_{x \rightarrow 0} f(x) - g(x) = \infty$
  - (c)  $\lim_{x \rightarrow 0} f(x) - g(x) = -\infty$
  - (d)  $\lim_{x \rightarrow 0} f(x) - g(x)$  does not exist.
5. Show that for any non-constant polynomial  $p(x)$  the limit of  $p(x)$  will be  $\infty$  or  $-\infty$  as  $x \rightarrow \infty$  or  $x \rightarrow -\infty$ .
6. Show that a rational function will have a finite nonzero limit at infinity if the degrees of the denominator and numerator are the same, and it will have a limit equal to zero if the degree of the numerator is smaller than the degree of the denominator.
7. Find the following limits (you may use the limit laws)

(a)  $\lim_{x \rightarrow 1^-} \frac{x}{x^3 - 1}$

(b)  $\lim_{x \rightarrow 3^+} \frac{x^2 - 2x}{3 - x}$

(c)  $\lim_{x \rightarrow -\infty} \frac{x^2}{x^3 + 3x - 2}$

(d)  $\lim_{x \rightarrow \infty} \frac{x^2 - 3x}{\sqrt{x^4 + 5x} - 2}$

(e)  $\lim_{x \rightarrow \infty} \sqrt{x - 2} - \sqrt{x - 4}$

Hint: You cannot use the limit laws immediately!