Second Midterm Exam - Solution

1. (a) Give the precise definition of the upper and lower integral of a function f on an interval $[a, b]$.

The lower integral is defined as

$$\underline{I}(f) = \sup \left\{ \int_a^b s(x)dx \middle| s(x) \leq f(x), \text{ for all } x \in (a,b), s(x) \text{ is a step function} \right\},$$

The upper integral is defined as

$$\overline{I}(f) = \inf \left\{ \int_a^b t(x)dx \middle| t(x) \geq f(x), \text{ for all } x \in (a,b), t(x) \text{ is a step function} \right\}.$$

(b) Explain, how one can show the integrability of a function f on an interval $[a, b]$ using the upper and lower integrals.

(Think about what the main idea was for the proof of the integrability of monotonic functions and continuous functions.)

We showed that if $\underline{I}(f) = \overline{I}(f)$ then the function is integrable, and the integral is equal to this value. We used this in the following way: for any $\varepsilon > 0$ we found step functions s, t so that $s(x) \leq f(x) \leq t(x)$ on $[a, b]$ and $\int_a^b t(x)dx - \int_a^b s(x)dx \leq \varepsilon$ which showed that $0 \leq \overline{I}(f) - \underline{I}(f) \leq \varepsilon$. Since this is true for every ε we get $\underline{I}(f) = \overline{I}(f)$.

2. Compute the following integrals:

(a) We have

$$|x^2 - 9| = \begin{cases} x^2 - 9 & x > 3, x \leq -3 \\ 9 - x^2 & -3 \leq x \leq 3 \end{cases}$$

Thus

$$\begin{aligned}
\int_{-2}^4 |x^2 - 9|dx &= \int_{-2}^3 |x^2 - 9|dx + \int_3^4 |x^2 - 9|dx \\
&= \int_{-2}^3 (9 - x^2)dx + \int_3^4 (x^2 - 9)dx \\
&= [9x - x^3/3]_{-2}^3 + [x^3/3 - 9x]_3^4 \\
&= ((27 - 27/3) - (-18 - (-8)/3)) + ((64/3 - 36) - (27/3 - 27)) \\
&= 110/3
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{-\pi/2}^{2\pi/3} \cos(2t)dt &= \frac{1}{2} \int_{-\pi}^{4\pi/3} \cos(t)dt \\
&= [\frac{1}{2}\sin(t)]_{-\pi}^{4\pi/3} = \frac{1}{2}(\sin(4\pi/3) - \sin(\pi)) = \frac{1}{2}(-\sqrt{3}/2 - 0) = -\frac{\sqrt{3}}{4}
\end{aligned}$$

In the first step we used the expansion-contraction properties of the integral.

3. Give an '$\varepsilon - \delta$' proof of the statement $\lim_{x\to -1}\sqrt{x+2} = 1$.

   Fix $\varepsilon > 0$, we need to find $\delta > 0$ so that $0 < |x+1| < \delta$ implies $|\sqrt{x+2}-1| < \varepsilon$. This inequality is equivalent to $1-\varepsilon < \sqrt{x+2} < 1+\varepsilon$. We may assume that $\varepsilon < 1$ and then the previous inequality is equivalent to

$$(1-\varepsilon)^2 - 1 < x+1 < (1+\varepsilon)^2 - 1$$
$$\varepsilon^2 - 2\varepsilon < x+1 < \varepsilon^2 + 2\varepsilon \qquad (*)$$

   Choosing $\delta = 2\varepsilon - \varepsilon^2 = \varepsilon(2-\varepsilon)$ (which is positive if $\varepsilon < 1$) will get the job done.

   Indeed, if $|x+1| < \delta$ then $\varepsilon^2 - 2\varepsilon < x+1 < 2\varepsilon - \varepsilon^2 < \varepsilon^2 + 2\varepsilon$ which means that inequality (*) holds. But then $|\sqrt{x+2}-1| < \varepsilon$ holds as well which is what we needed.

4. Find the following limits. (You can use anything we proved in class about limits, but you need to explain your steps!)

   (a)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(5x)}{\sqrt{1+x}-\sqrt{1-x}} &= \lim_{x\to 0}\frac{\sin(5x)(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}\\ &= \lim_{x\to 0}\frac{\sin(5x)(\sqrt{1+x}+\sqrt{1-x})}{2x}\\ &= \lim_{x\to 0}\frac{\sin(5x)}{5x}\frac{(\sqrt{1+x}+\sqrt{1-x})}{2/5}\end{aligned}$$

   The function $\frac{(\sqrt{1+x}+\sqrt{1-x})}{2/5}$ is continuous at zero so its limit is $\frac{(\sqrt{1+0}+\sqrt{1-0})}{2/5} = 5$. The limit of $\frac{\sin(5x)}{5x}$ is the same as $\frac{\sin(x)}{x}$ as $x \to 0$ which is 1. Using the fact that the limit of products is the product of limits we get that

$$\lim_{x\to 0}\frac{\sin(5x)}{\sqrt{1+x}-\sqrt{1-x}} = 5.$$

   (b)

$$\lim_{x\to\infty}\frac{2}{3x+5} = \lim_{x\to\infty}\frac{2/x}{3+5/x}$$

   Since $\lim_{x\to\infty} 1/x = 0$ we have $\lim_{x\to\infty} 2/x = 0$ and $\lim_{x\to\infty} 3+5/x = 3$ (using the sum and product rule of limits). Then the limit of the ratio is the ratio of the limits (since the limit of the denominator is not zero), so $\lim_{x\to\infty}\frac{2}{3x+5} = 0$.

   Another way to prove this would be by the application of the squeezing principle. If $x > 0$ then

$$0 < \frac{2}{3x+5} < \frac{2}{3x}.$$

   Since $1/x \to 0$ as $x \to \infty$, this will also be true for $\frac{2}{3}\frac{1}{x}$ (by the product rule of limits). Another proof would be by using the $\varepsilon - \delta$ definition (with a large constant c in place of δ, since we take the limit at ∞).

5. Suppose that the function $f(x)$ is continuous on the *open* interval $(0,1)$, and we also have $\lim_{x\to 0^+} f(x) = 0$, $\lim_{x\to 1^-} f(x) = 1$.

   (a) (Show that there is a continuous function $g(x)$ on $[0,1]$ for which $g(x) = f(x)$ for every $x \in (0,1)$.

   Define the function $g(x)$ on $[0,1]$ as follows:

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

   This is continuous for every $x \in (0,1)$ (since it's equal to f which is continuous by assumption). It is also right-continuous at $x = 0$ (since $\lim_{x\to 0^+} f(x) = 0 = g(0)$) and the same is true for $x = 1$. But this means that g is continuous on $[0,1]$.

   (b) Show that f is bounded in $(0,1)$.

   The function g is continuous on $[0,1]$ so by the Extreme Value Theorem it is bounded there: $|g(x)| < M$ for some M for $x \in [0,1]$. But then $|f(x)| < M$ is true for all $x \in (0,1)$ since $g(x) = f(x)$ there.

   (c) Is it always true that f has an absolute maximum in $(0,1)$? If yes, prove it, if not then give a counterexample.

   No. E.g. $f(x) = x$ is a function like this and it does not have an absolute maximum in $(0,1)$. (Since it's strictly less than 1 and it can be bigger than $1-\varepsilon$ for any $\varepsilon > 0$.)

6. Show that the equation $\sqrt{x} = \cos(x^3)$ has at least one real solution.

   The function $f(x) = \sqrt{x} - \cos(x^3)$ is continuous for every $x \geq 0$ (right-continuous at $x = 0$). We have $f(0) = -1$ and $f(4) = \sqrt{4} - \cos(64) \geq 2 - 1 = 1$. Thus $f(0) < 0, f(4) > 1$ and by Bolzano's theorem there must be an $x \in (0,4)$