First Midterm Exam – Solutions

1. (15 pts) Decide for each of the following statements if they are true or not. If yes, give a short proof why. If not, give a counterexample.

   (a) The sum of two rational numbers is rational.

   TRUE. Let q, r be rational. Then there exist integers a, b, c, d with $b, d \neq 0$ such that $q = a/b$ and $r = c/d$. Then $q + r = \frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$. Since $ad + cb$ and bd are integers we know $q + r$ is a rational number.

   (b) The product of two irrational numbers is irrational.

   FALSE. E.g. take $\sqrt{2} \cdot (-\sqrt{2}) = -2$.

   (c) A bounded set of real numbers always contains a maximal element.

   FALSE. E.g. $S = (0, 1)$ is bounded but $\sup S = 1 \notin S$.

2. (20 pts)

   (a) Give the precise definition of the supremum of a set of real numbers.

   A number B is called a supremum of a nonempty set S of real numbers if B has the following two properties:
   a) B is an upper bound for S.
   b) No number less than B is an upper bound for S.

   (b) State the least upper bound axiom (with all the appropriate conditions).

   Every nonempty set S of real numbers that is bounded above has a supremum.

3. (20 pts) Show that for any positive integer n we have

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$$

   Hint: use induction.

   Base case: $n = 1$. We have $1 \leq 2$.

   Induction step:
   Assume $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$, want to show $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}$.
   We know by the induction hypothesis that $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}}$.
   Now we need to check that $2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}$.

Since everything is positive we can square both sides and this is equivalent to checking

$$4 + \frac{4\sqrt{n}}{\sqrt{n+1}} + \frac{1}{n+1} \leq 4(n+1).$$

The left hand side yields

$$4 + \frac{4\sqrt{n}}{\sqrt{n+1}} + \frac{1}{n+1} = \frac{4n + 4 + 4\sqrt{n^2+n}}{n+1}.$$

So we check that $4n + 4 + 4\sqrt{n^2+n} \leq 4(n+1)^2$.

We know that $n^2 + n \leq n^2 + 2n + 1$, thus

$$4n + 4 + 4\sqrt{n^2+n} \leq 4n^2 + 4n + 4(n+1) = 4(n+1)^2.$$

□

4. (25 pts) Let f and g be step functions defined the following way:

$$f(x) = \begin{cases} 1 & \text{if } 2 \leq x < 3, \\ 2 & \text{if } 3 \leq x < 5, \\ 3 & \text{if } 5 \leq x \leq 8, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2 & \text{if } 1 \leq x < 4, \\ -1 & \text{if } 4 \leq x < 6. \end{cases}$$

(a) Give a short definition of a step function.

A function s, whose domain is a closed interval $[a, b]$, is called a step function if there is a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ such that s is constant on each open subinterval of P. That is to say, for each $k = 1, 2, ..., n$, there is a real number s_k such that

$$s(x) = s_k \text{ if } x_{k-1} < x < x_k.$$

Note: the textbook insists on using closed intervals in the definition, but this is not taken that seriously in practice.

(b) Show that $f + g$ is a step function and describe it fully! (Be careful with the domain!)

Note that f is deifned on $[2, 8]$ and g is defined on $[1, 6)$. So $f + g$ can only be defined on $[2, 6)$. Thus strictly speaking as a step function $f + g$ is only defined on $[2, 6)$. (If we take the strict definition with the closed intervals then this wouldn't even be a step-function.) We have

$$f + g = \begin{cases} 3 & x \in [2, 3) \\ 4 & x \in [3, 4) \\ 1 & x \in [4, 5) \\ 2 & x \in [5, 6) \end{cases}$$

(c) Evaluate the integral $\int_2^5 (f(x) + g(x))dx$.

Using part b) we get $\int_2^5 (f(x) + g(x))dx = 3 + 4 + 1 = 8$.

5. (20 pts)

(a) Recall that the conjugate of a complex number $z = a+bi$ (with $a, b \in \mathbb{R}$) is defined as $\bar{z} = a - bi$. Prove that for any two complex numbers z_1, z_2 we have $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ and $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$.

Let $z_1 = a_1 + b_1 i,\ z_2 = a_2 + b_2 i$. Then

$$z_1 + z_2 = a_1 + b_1 i + a_2 + b_2 i = (a_1 + a_2) + i(b_1 + b_2).$$

So $\overline{z_1 + z_2} = (a_1 + a_2) - i(b_1 + b_2)$.

Also $\bar{z_1} = a_1 - b_1 i$ and $\bar{z_2} = a_2 - b_2 i$. Thus

$$\bar{z_1} + \bar{z_2} = (a_1 + a_2) - i(b_1 + b_2) = \overline{z_1 + z_2}.$$

Similarly we have

$$z_1 \cdot z_2 = a_1 a_2 + a_1 b_2 i + a_2 b_1 i - b_1 b_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1).$$

So $\overline{z_1 \cdot z_2} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1)$.

And we have $\bar{z_1} \cdot \bar{z_2} = (a_1 - b_1 i)(a_2 - b_2 i) = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1)$.

Thus $\overline{z_1 \cdot z_2} = \bar{z_1} \cdot \bar{z_2}$.

(b) Show that if the complex number z is a solution of $z^6 + 3z^5 - 4z^4 + 8z^2 - z + 1 = 0$ then $\bar{z}$ is also a solution.

Hint: You cannot solve the equation explicitly. You have to use the properties of the $z \to \bar{z}$ function to solve the problem. (If you are doing complicated computations then you are not on the right track...)

This means we must show $\bar{z}^6 + 3\bar{z}^5 - 4\bar{z}^4 + 8\bar{z}^2 - \bar{z} + 1 = 0$ Thus it suffices to show that

$$\overline{z^6 + 3z^5 - 4z^4 + 8z^2 - z + 1} = \bar{z}^6 + 3\bar{z}^5 - 4\bar{z}^4 + 8\bar{z}^2 - \bar{z} + 1$$

as $\bar{0} = 0$. However this follows from repeated application of part a) because we know that $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ and $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$. Also from class we know that for $a \in \mathbb{R}$ and $z \in \mathbb{C}$ we have that $\bar{a} = a$ and $\overline{a \cdot z} = a\bar{z}$.