Ex 2.25

① LB G open. Let $x \in G$.

$$\underline{\lim} \frac{1}{r_n} \log \mu_n(G) \geq -\underline{\kappa}(x) = -\kappa(x).$$

def of $\underline{\kappa}$

Take sup over x on the right.

② UB F compact. Let $c < \inf_{x \in F} \kappa(x)$.

$\forall x \in F$ pick open $G_x \ni x$ s.t. $-\overline{\lim} \frac{1}{r_n} \log \mu_n(G_x) > c$.

(By the def. of $\overline{\kappa}(x)$.)

Cover F with $G_{x_1} \cup \dots \cup G_{x_m}$, by compactness.

$$\overline{\lim} \frac{1}{r_n} \log \mu_n(F) \leq \overline{\lim} \frac{1}{r_n} \log \sum_{i=1}^{m} \mu_n(G_{x_i})$$

$$\leq \bigvee_{1 \leq i \leq m} \overline{\lim} \frac{1}{r_n} \log \mu_n(G_{x_i}) < -c.$$

Let $c \nearrow \inf_F \kappa$.

③ LSC'y: $\kappa(x) > c \Rightarrow \underline{\kappa}(x) > c \Rightarrow -\underline{\kappa}(x) < -c$

$\Rightarrow \exists$ open $G \ni x$ s.t. $\underline{\lim} \frac{1}{r_n} \log \mu_n(G) < -c$

$\Rightarrow -\underline{\kappa}(y) < -c \quad \forall y \in G$

$\Rightarrow \kappa(y) > c \quad \forall y \in G$. Hence $x \in G \subseteq \{\kappa > c\}$.

The set $\{\kappa > c\}$ is open.

$\boxed{\text{Ex. } 3.9}$

<u>UB</u> A closed, $\bar{f}(x) = \begin{cases} f(x), & x \in A \\ -\infty, & x \notin A \end{cases}$

$\bar{f}$ is usc because $\{\bar{f} < c\} = \{f < c\} \cup A^c$ is open and $\bar{f}$ is bounded above because f is.

By the UB lemma for Varadhan's thm,

$$\overline{\lim} \frac{1}{r_n} \log \nu_n(A) = \overline{\lim}\left(\frac{1}{r_n} \log \int e^{r_n \bar{f}} d\mu_n - \frac{1}{r_n} \log \int e^{r_n f} d\mu_n\right)$$

$$\leq \underbrace{\sup_{\bar{f} \wedge I < \infty} (\bar{f} - I)}_{\boxed{= \sup_{x \in A} (f(x) - I(x))}} - \sup (f - I)$$

Note that Varadhan's thm applies to f and the limit $\sup(f - I)$ is finite.

$$= -\inf_{x \in A} \{ I(x) - f(x) - \inf (I - f)\}$$

<u>LB</u> G open, $\underline{f}(x) = \begin{cases} f(x), & x \in G \\ -\infty, & x \notin G \end{cases}$ is lsc because $\{\underline{f} > c\} = \{f > c\} \cap G$ is open.

By the LB lemma for V's thm,

$$\underline{\lim} \frac{1}{r_n} \log \nu_n(G) = \underline{\lim}\left(\frac{1}{r_n} \log \int e^{r_n \underline{f}} d\mu_n - \frac{1}{r_n} \log \int e^{r_n f} d\mu_n\right)$$

$$\geq \underbrace{\sup_{\underline{f} \wedge I < \infty} (\underline{f} - I)}_{= \sup_G (f - I)} - \sup (f - I)$$

$$= -\inf_{x \in G} \{ I(x) - f(x) - \inf (I - f)\}.$$