Math 635 Introduction to Stochastic Calculus, Spring 2014
Homework 6

Due 3 PM on Wednesday, April 16.

Generalities. Throughout these exercises $B_\cdot = \{B_t : t \in \mathbb{R}_+\}$ is standard Brownian motion.

1. (a) One of the basic things we showed was the Itô isometry

$$E\Big[\Big(\int_0^t f(s)\,dB_s\Big)^2\Big] = E\int_0^t f(s)^2\,ds$$

for $f \in \mathcal{H}^2[0,t]$. Use this to derive the identity

$$E\Big[\int_0^t f(s)\,dB_s \cdot \int_0^t g(s)\,dB_s\Big] = E\int_0^t f(s)g(s)\,ds$$

for $f, g \in \mathcal{H}^2[0,t]$.

(b) Let

$$U(t) = \begin{bmatrix} u_{11}(t) & u_{12}(t) \\ u_{21}(t) & u_{22}(t) \end{bmatrix}$$

be a bounded measurable 2×2-matrix valued function on $[0,T]$. Let $\vec{B}_t$ be a standard 2-dimensional Brownian motion, and define

$$\vec{X}_t = \begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix} = \int_0^t U(s)\,d\vec{B}_s.$$

Use part (a) to find the 2×2 covariance matrix of the random vector $\vec{X}_t$ for a fixed time t. Express it in a compact form utilizing the matrix $U(s)$.

2. Find a deterministic function $\tau : [0,\infty) \to [0,\infty)$ such that

$$X_t = \int_0^t e^s\,dB_s \qquad \text{and} \qquad Y_t = B_{\tau(t)}$$

have the same distribution as processes. *Hint:* Proposition 7.6 points in the right direction.

3. Let $B_1(t), B_2(t), B_3(t), \ldots$ be independent standard Brownian motions. For $k \in \mathbb{N}$ let

$$A_k(t) = \int_0^t g_k(B_1(s), B_2(s), \ldots, B_k(s))\, ds$$

for some function g_k. Find processes A_2 and A_3 of this type such that

$$B_1(t)^2 B_2(t)^2 - A_2(t) \quad \text{and} \quad B_1(t)^2 B_2(t)^2 B_3(t)^2 - A_3(t)$$

are martingales. Do not forget to verify everything that is needed to establish that these processes are martingales.

4. Problem 9.3 from p. 150 of the book. Ignore the last question about comparison with Brownian bridge. (I don't know what the question means, but if you can see some point in it, please write it down.) You can follow the book's suggestion or use an integrating factor.

5. Find a solution Y_t to the SDE

$$dY_t \;=\; r\, dt \;+\; \alpha Y_t\, dB_t$$

with a given initial value Y_0 independent of the Brownian motion. Check your solution by using Itô's formula to compute dY_t. *Hint.* You may need some trial and error to find the right integrating factor.