Math 635 Introduction to Stochastic Calculus, Spring 2014
Homework 5

Due 3 PM on Friday, April 4.

Generalities. Throughout these exercises $B_\cdot = \{B_t : t \in \mathbb{R}_+\}$ is standard Brownian motion.

1. Let $f \in L^1[0,T]$ and B_t standard Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Define

$$Y_f(\omega) = \int_0^T f(s) B_s(\omega)\, ds.$$

(a) Identify the distribution of the random variable Y_f with the help of Itô's formula. *Hint.* Apply integration by parts to $\int_0^T F(s)\, dB_s$ where $F(t) = \int_0^t f(s)\, ds$. Note that $F(t)B_t = \int_0^t F(t)\, dB_s$. Use Prop. 7.6 from p. 101.

(b) For two functions $f, g \in L^1[0,T]$, identify the joint distribution of the vector $\begin{bmatrix} Y_f \\ Y_g \end{bmatrix}$.

2. Let $\delta, \mu \in \mathbb{R}$, $X_t = \mu t + B_t$ and

$$Y_t = \int_0^t e^{\delta(X_t - X_s) - \frac{1}{2}\delta^2 (t-s)}\, ds.$$

Show that the process Y_t satisfies the equation

$$Y_t = \int_0^t (1 + \delta \mu Y_s)\, ds + \delta \int_0^t Y_s\, dB_s.$$

3. Fix $0 < T < \infty$. Show that for almost every ω,

$$\lim_{\lambda \to \infty} \sup_{t \in [0,T]} \left| e^{-\lambda t} \int_0^t e^{\lambda s}\, dB_s(\omega) \right| = 0.$$

Hint. There are probably several ways of doing this. One way is to rewrite the integral suitably and then just use continuity of Brownian motion.