Math 635 Introduction to Stochastic Calculus, Spring 2014
Homework 1

Due 3 PM on Monday, February 3

1. Give an example of a sequence of random variables X_n and a random variable X, all defined on the same probability space $(\Omega, \mathcal{F}, P)$, and such that $X_n(\omega) \to X(\omega)$ for each $\omega \in \Omega$ but $EX_n \to EX$ fails. (Limits as $n \to \infty$.) Note that you need to find an example that violates the hypotheses of DCT and MCT (p. 278 in the book).

2. Let $\Omega = \mathbb{R}$ and on this Ω put the probability measure P defined for Borel sets $A \subseteq \mathbb{R}$ by

$$P(A) = \frac{\lambda(A \cap [-10, 10])}{20}$$

where λ is Lebesgue measure on $\mathbb{R}$. In words, P is the uniform probability measure on the interval $[-10, 10]$. Examples of values of P are $P([a, b]) = (b - a)/20$ for $[a, b] \subseteq [-10, 10]$, $P([-100, 0]) = 1/2$, and $P(A) = 0$ if A lies outside $[-10, 10]$.

Define the random variable (measurable function) $X : \Omega \to \mathbb{R}$ by

$$X(\omega) = \begin{cases} -5 & \text{if } \omega \le -5 \\ \omega & \text{if } -5 < \omega \le 0 \\ k & \text{if } \omega \in (k-1, k] \text{ for a positive integer } k. \end{cases}$$

Note that X is not purely discrete, nor does it have a density.

(a) Find the probabilities $P(X \le -7)$, $P(X \le -4)$, $P(X \le 12)$, and $P(-2 \le X \le 3.5)$.

(b) Compute the expectation $E(X)$. *Hint.* Evaluate the Lebesgue integral $E(X) = \int_\Omega X(\omega)\, P(d\omega)$ piece by piece. Use some common sense and some calculus.

(c) Define the subsets $A = \{-8, 0, 1, 2\}$ and $B = (5, 12]$ of Ω. Decide whether A and B are members of the σ-algebra $\sigma(X)$ on Ω generated by X. Is there a Borel subset of $(-5, 0]$ that is not a member of $\sigma(X)$? (Note that part (b) does not involve the measure P at all, only properties of X as a function.)

3. (Exercise 4.1(a) in the book: tower property) Use the definition of conditional expectation to prove that if $\mathcal{H}$ is a sub-σ field of $\mathcal{G}$ then

$$E[E(X|\mathcal{G})|\mathcal{H}] = E(X|\mathcal{H}).$$

4. Suppose X has Exp(λ) distribution. This is the exponential distribution with parameter λ, which means that X has density $f(x) = \lambda e^{-\lambda x}$ on $[0, \infty)$, and $f(x) = 0$ on $(-\infty, 0)$.

A possible way to realize a probability space $(\Omega, \mathcal{F}, P)$ for X is to take $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}_{\mathbb{R}}$, and let the measure P be defined for Borel sets B by

$$P(B) = \int \mathbf{1}_B(x) f(x)\, dx$$

where the integral is interpreted as a Lebesgue integral over $\mathbb{R}$. For intervals this gives the familiar values $P((a, b]) = F(b) - F(a)$ where F is the c.d.f. defined by

$$F(x) = \int_{-\infty}^{x} f(y)\, dy.$$

On this Ω, define $X(\omega) = \omega$. On this same probability space define a random variable Y by

$$Y(\omega) = \begin{cases} 3, & \omega \le 10, \\ 27, & \omega > 10. \end{cases}$$

Find the random variable $E(X|Y)(\omega)$.

5. Suppose (X, Y) is a pair of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose (X, Y) has a joint density $f(x, y)$. This means that f is a nonnegative function on $\mathbb{R}^2$ that satisfies

$$E[H(X, Y)] = \iint_{\mathbb{R}^2} H(x, y) f(x, y)\, dx\, dy$$

for any bounded Borel measurable function $H : \mathbb{R}^2 \to \mathbb{R}$. The marginal density f_Y of Y is defined by

$$f_Y(y) = \int_{\mathbb{R}} f(x, y)\, dx.$$

In an elementary probability course we define the conditional density of X, given that $Y = y$, by

$$f(x|y) = \begin{cases} \dfrac{f(x,y)}{f_Y(y)} & \text{if } f_Y(y) > 0 \\ 0 & \text{if } f_Y(y) = 0. \end{cases}$$

Let ϕ be a bounded Borel function on $\mathbb{R}$ and define

$$Z(\omega) = \int_{\mathbb{R}} \phi(x)\, f(x|Y(\omega))\, dx.$$

Show that Z is the conditional expectation $E[\phi(X)|Y]$. The measurability issue is immediate as Z is a function of Y. For the other part of the definition of conditional expectation, you need to check that

$$E[Z\,\psi(Y)] = E[\phi(X)\,\psi(Y)]$$

for an arbitrary bounded Borel function ψ. To check this, evaluate the expectations by integrating over $\mathbb{R}^2$ with the help of the densities.