Math 629 Introduction to Measure and Integration Spring 2006
Homework 6

Due Friday, April 14

When nothing else is specified, the exercise takes place on some abstract measure space $(X, \mathcal{M}, \mu)$.

1. Let $0 < \rho < 1$. Define a Borel probability measure μ on $\mathbb{R}$ by

$$\mu(B) \;=\; \tfrac{1}{2}\int_{\mathbb{R}} \mathbf{1}_B(x)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \;+\; \tfrac{1}{2}\sum_{k\in\mathbb{N}} \rho(1-\rho)^{k-1}\mathbf{1}_B(k)\,, \quad B \in \mathcal{B}_{\mathbb{R}}\,,$$

and let X be a random variable with distribution μ. The dx integral above stands for an integral with Lebesgue measure. (Probabilistically speaking, X follows a standard normal distribution with probability 1/2, and a geometric distribution with parameter ρ with probability 1/2.)

(a) Show that for all $f \in L^1(\mu)$,

$$\int_{\mathbb{R}} f\,d\mu \;=\; \tfrac{1}{2}\int_{\mathbb{R}} f(x)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \;+\; \tfrac{1}{2}\sum_{k\in\mathbb{N}} \rho(1-\rho)^{k-1}f(k).$$

Hint: start with $f = \mathbf{1}_B$.

(b) Show that X is integrable. (Integrability means $E\,|X| < \infty$.)

(c) Compute the mean $E(X)$.

Justify all the steps you take rigorously.

2. Exercise 19 on page 59. *Uniform convergence* means this: $f_n \to f$ uniformly if for every $\varepsilon > 0$ there exists $N < \infty$ such that for all $n \geq N$,

$$\sup_{x\in X} |f_n(x) - f(x)| \leq \varepsilon.$$

In other words, the point is that the *same* N works for *all* x once ε is given. In pointwise convergence different x might need a different N. So quite obviously uniform convergence implies pointwise convergence.

3. Exercise 48 on page 69. What is the significance of this exercise to Fubini's theorem?