Math 629 Introduction to Measure and Integration Spring 2006
Homework 4

Due Wednesday, March 22

When nothing else is specified, the exercise takes place on some abstract measure space $(X, \mathcal{M}, \mu)$. A series $\sum x_k$ of real or complex numbers converges *absolutely* if $\sum |x_k| < \infty$. This implies the series has a limit $x = \sum x_k$ in the usual sense of convergence of partial sums.

1. (a) To put one source of confusion to rest, devise an example of a measurable function f and a measurable set A such that $f(A)$ is not measurable. (The point being that *inverse images* of measurable sets under measurable functions are measurable, but *images* can fail to be measurable.) Your space does not need more than two points.

(b) For any quantities $b_k \in [0, \infty]$, the value of the series $\sum b_k$ is well-defined in $[0, \infty]$ as the monotone limit of partial sums $\sum_{k=1}^{n} b_k$ as $n \to \infty$ (just stating something we know already). Prove that if $a_k \in [0, \infty]$, $\{E_k\}$ are measurable sets, and we define

$$f(x) = \sum_{k=1}^{\infty} a_k \mathbf{1}_{E_k}(x),$$

then $f \in L^+$ and

$$\int_X f \, d\mu = \sum_{k=1}^{\infty} a_k \mu(E_k).$$

2. Suppose $\{a_k\}$ is a sequence of real numbers and $\{c_k\}$ a sequence of nonnegative numbers that satisfy $\sum |a_k| \, c_k < \infty$. Suppose $\{E_k\}$ are measurable sets in a measure space $(X, \mathcal{M}, \mu)$ such that $c_k = \mu(E_k)$. We would like to define a function g by the series

$$g(x) = \sum_{k=1}^{\infty} a_k \mathbf{1}_{E_k}(x) \tag{1}$$

but a priori we do not know about the convergence of this series.

(a) Show that the series defining g converges absolutely almost everywhere. Using this, give a rigorous definition of some measurable function h defined on the whole space X that agrees with formula (1) a.e. Hint: We proved in class a useful consequence of $\int f < \infty$ for $f \in L^+$.

(b) Now that we can regard g as a measurable function (defined at least a.e.), show that g is integrable and

$$\int_X g \, d\mu = \sum_{k=1}^{\infty} a_k \mu(E_k).$$

3. Let m denote Lebesgue measure on the Borel subsets of $X = [0, 1]$, and $f(x) = x$. Use the approximation results of Section 2.2 to compute the integral $\int_X f \, dm$. You should not resort to calculus to do the exercise, but your result should agree with what calculus gives.

4. Exercise 13 on page 52.