Math 629 Introduction to Measure and Integration Spring 2006
Homework 3

Due Friday, March 3

1. A *filtration* is a nested, indexed family of σ-algebras $\{\mathcal{F}_t : 0 \le t < \infty\}$ on some space Ω that satisfy $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s < t$. Filtrations arise whenever time-indexed stochastic processes are used. For example, in an application $\mathcal{F}_t$ could represent all the information about a stock price up until time t. (At this point it is probably not clear how σ-algebras represent information. We will discuss this at a later time in conjunction with stochastic processes.)

Show that the union $\bigcup_{0\le t<\infty} \mathcal{F}_t$ is an algebra, and show by example that it is not necessarily a σ-algebra.

2. This exercise walks you through an issue which has some consequences for the theory of stochastic processes.

(a) Let $\{(Y_\alpha, \mathcal{K}_\alpha) : \alpha \in J\}$ be a family of measurable spaces, X a space, and suppose for each $\alpha \in J$ there is a function $f_\alpha : X \to Y_\alpha$. Here J is some arbitrary index set.

For each $I \subseteq J$, let $\mathcal{G}_I$ be the σ-algebra on X generated by the functions $\{f_\alpha : \alpha \in I\}$. Equivalently,

$$\mathcal{G}_I = \sigma\{f_\alpha^{-1}(E_\alpha) : \alpha \in I, E_\alpha \in \mathcal{K}_\alpha\}.$$

Let

$$\mathcal{H} = \{H \subseteq X : \text{ there is a countable subset } I \text{ of } J \text{ such that } H \in \mathcal{G}_I \}.$$

Show that $\mathcal{H}$ is a σ-algebra, and then that $\mathcal{G}_J = \mathcal{H}$.

(b) With everything as in part (a), show that if $x \in A$, $A \in \mathcal{G}_I$ and $f_\alpha(x) = f_\alpha(y)$ for all $\alpha \in I$, then also $y \in A$. Hint: Show that the collection

$$\mathcal{B} = \{H \subseteq X : \text{if } f_\alpha(x) = f_\alpha(y) \text{ for all } \alpha \in I, \text{ then either } \{x, y\} \subseteq H \text{ or } \{x, y\} \subseteq H^c \}$$

is a σ-algebra.

(c) Now let $\Omega = \mathbb{R}^{[0,1]}$, the space of all functions $\omega : [0,1] \to \mathbb{R}$. Define the projection mappings $\pi_t : \Omega \to \mathbb{R}$ by $\pi_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \in [0,1]$. Let $\mathcal{F}$ be the σ-algebra on Ω generated by the projections $\{\pi_t : t \in [0,1]\}$, in other words

$$\mathcal{F} = \sigma\{\pi_t^{-1}(B) : B \in \mathcal{B}_{\mathbb{R}}, t \in [0,1]\}.$$

So Ω is a Cartesian product space and $\mathcal{F}$ is the product σ-algebra on it.

Let $C \subseteq \Omega$ be the subset of continuous functions,

$$C = \{\omega \in \Omega : \ \omega \text{ is a continuous function from } [0,1] \text{ into } \mathbb{R} \ \}.$$

Show that $C \notin \mathcal{F}$.

Remark. The point of this last exercise is that two very natural things are not compatible: the product σ-algebra is the natural σ-algebra to put on a product space, while continuity is certainly a natural property of functions that we study. But if this Ω is the sample space for a probability model whose outcome is a function (for example the random path of a particle moving in space), this exercise suggests that we cannot even ask the question "what is the probability that the random path is continuous" because the event in question is not measurable. The way around this difficulty is to construct the probability model on some other space, and not use $\mathbb{R}^{[0,1]}$. This issue is dealt with in textbooks that cover the theory of stochastic processes in continuous time.