Math 629 Introduction to Measure and Integration Spring 2006
Homework 1

Due Monday, Feb. 6

Definition: the *indicator function* $\mathbf{1}_A$ of a set A is defined by

$$\mathbf{1}_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

1. Let $A_1, A_2, A_3, \dots$ be a sequence of subsets of some underlying space X, and $A = \bigcup_{k\geq 1} \bigcap_{n \geq k} A_n$. Using the definition of liminf on p. 11, show that for each $x \in X$,

$$\mathbf{1}_A(x) = \varliminf_{n\to\infty} \mathbf{1}_{A_n}(x).$$

2. Let us define the Borel σ-algebra $\mathcal{B}_{\overline{\mathbb{R}}}$ of the extended real line as the collection of sets

$$\{G, G \cup \{\infty\}, G \cup \{-\infty\}, G \cup \{-\infty, \infty\} : G \in \mathcal{B}_{\mathbb{R}}\}.$$

Show that $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by the collection $\{[q, \infty] : q \in \mathbb{Q}\}$ but not by $\{(a, b) : \ a < b,\ a \text{ and } b \text{ are real}\}$. Here by $[q, \infty]$ is meant

$$[q, \infty] = \{x \in \mathbb{R} : q \leq x < \infty\} \cup \{\infty\}.$$

(You are free to use Proposition 1.2.)

3. Exercise 4 on p. 24.