Math 629 Introduction to Measure and Integration Spring 2006
Take-Home Final Exam

Due Monday, May 8, 4 PM, in the instructor's office VV419. Absolutely no collaboration permitted. If you need to discuss the exam, see the instructor. 30 pts total.

Some definitions and notation. For a set B on the real line, $B - x = \{y - x : y \in B\}$. $\mathcal{B}_{\mathbb{R}}$ denotes the Borel σ-algebra on $\mathbb{R}$. If Z is a real-valued random variable on $(\Omega, \mathcal{F}, P)$, the distribution μ of Z is the probability measure $\mu(B) = P\{Z \in B\}$, $B \in \mathcal{B}_{\mathbb{R}}$.

1. Let μ, ν be Borel probability measures on $\mathbb{R}$.

(a) (4 pts) Let $B \in \mathcal{B}_{\mathbb{R}}$. Show that the function $g(x) = \mu(B - x)$ is Borel measurable. (Hint: Reasoning via a product space might be helpful. Start by writing $\mu(B - x)$ as an integral. Notice that $y \in B - x$ iff $x + y \in B$.)

(b) (4 pts) Show that

$$\alpha(B) = \int_{\mathbb{R}} \mu(B - x)\, \nu(dx)$$

defines a probability measure α on the Borel σ-algebra of $\mathbb{R}$. The measure α is called the *convolution* of μ and ν, and denoted by $\mu * \nu$. (Part (a) assures us that the integration makes sense. Here one needs to check the properties of a probability measure.)

(c) (3 pts) Show that $\mu * \nu = \nu * \mu$.

(d) (4 pts) Let X and Y be two independent real-valued random variables on $(\Omega, \mathcal{F}, P)$ such that X has distribution μ and Y has distribution ν. Show that the random variable $X + Y$ has distribution $\mu * \nu$.

2. (5 pts) Suppose $\{X_n\}$, X are real-valued random variables such that $X_n \to X$ in probability. (Means the same as $X_n \to X$ in measure.) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then also $f(X_n) \to f(X)$ in probability. (Hint: recall the connection with almost everywhere convergence.)

Give an example of a function f and random variables $\{X_n\}$, X for which the convergence $f(X_n) \to f(X)$ (in prob) fails even though $X_n \to X$ (in prob).

3. (5 pts) For $i = 1, 2$ let ν_i and μ_i be σ-finite positive measures on the measurable space $(X_i, \mathcal{M}_i)$. Assume $\nu_i \ll \mu_i$ with Radon-Nikodym derivative $f_i(x_i) = \frac{d\nu_i}{d\mu_i}(x_i)$. Show that then $\nu_1 \otimes \nu_2 \ll \mu_1 \otimes \mu_2$ and

$$\frac{d(\nu_1 \otimes \nu_2)}{d(\mu_1 \otimes \mu_2)}(x_1, x_2) = f_1(x_1) f_2(x_2).$$

4. (5 pts) Let $(X, \mathcal{M}, \mu)$ be a measure space, μ a positive measure. Suppose $f_n : X \to [-\infty, \infty]$ and $g : X \to [0, \infty]$ are measurable, $\int g \, d\mu < \infty$, and $f_n \geq -g$ for all n. Show that

$$\int (\liminf f_n) \, d\mu \leq \liminf \int f_n \, d\mu.$$

(In other words, the conclusion of Fatou's Lemma is valid.) Be sure to justify cancellations properly.

Give an example of a sequence of real-valued functions $\{f_n\}$ for which the conclusion is not valid.