Math 521 Homework #9

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Chapter 6. Problem 2.

Proof. Proof by contradiction. Assume that there exists a number x_0 in $[a, b]$, such that $f(x_0) > 0$.

Choose $\epsilon = \frac{1}{2}f(x_0) > 0$, because f is continuous on $[a, b]$, we know that there exists a $\delta > 0$, such that for all $x \in [a, b] \cap (x_0 - \delta, x_0 + \delta)$, $f(x) > \frac{1}{2}f(x_0) > 0$.

Let $I = [a, b] \cap (x_0 - \delta, x_0 + \delta)$ and $l =$ the length of I. We know that $l > 0$ and can find a partition $P_0 = (x_0 = a, x_1, \cdots, x_n = b)$ of $[a, b]$, such that P_0 contains I. Given that $f(x) > \frac{1}{2}f(x_0)$ on I, we know $inf(f(x)) \geq \frac{1}{2}f(x_0)$ on I. Then, by definition of lower sum,

$$L(P_0, f) = \sum_{i=1}^{n} m_i \Delta x_i \geq \frac{1}{2}f(x_0)l > 0$$

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Because $\int_a^b f(x)dx = supL(P, f)$, for all P of $[a, b]$, we can get

$$\int_a^b f(x)dx \geq L(P_0, f) > 0$$

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This contradicts with $\int_a^b f(x)dx = 0$. □

Chapter 6 Problem 4.

Proof. Given that rational and irrational are dense in $\mathbb{R}$, for any partition $P = (x_0 = a, x_1, \cdots, x_n = b)$ of $[a, b]$, each interval $[x_{i-1}, x_i]$ contains both rational and irrational. Then, by definition, all $L(P, f) = 0$, and all $U(P, f) = b - a$. Thus, we have

$$supL(P, f) \neq infU(p, f)$$

By definition of integral, we know that $\int_a^b f(x)dx$ does not exist. □

Chapter 6. Problem 8.

Proof. First, we show that if $\int_1^\infty f(x)dx$ converges, then $\sum_1^\infty f(n)$ converges.

Because $\int_1^\infty f(x)dx$ converges, by definition, we know that there exists a number l, such that $l < \infty$ and $\int_1^\infty f(x)dx = l$.

By properties of integral, we know that for any $N \geq 1$ and $N \in \mathbb{N}$, we have

$$\int_1^\infty f(x)dx = \int_1^N f(x)dx + \int_N^\infty f(x)dx$$

Given that $f(x) \geq 0$, we know $\int_N^\infty f(x)dx \geq 0$. Thus, we have

$$\int_1^\infty f(x)dx = \int_1^N f(x)dx + \int_N^\infty f(x)dx \geq \int_1^N f(x)dx$$

Let $P_0 = (x_0 = 1, \cdots, x_{N-1} = N)$ (each $x_i = i + 1$) be a partition of $[1, N]$. Because f monotonically decreases on $[1, \infty)$, we know that in any subinterval $[n-1, n]$ of P_0, let $m_{n-1} = inf(f(x))$ on $[n-1, n]$, then $m_{n-1} = f(n)$. Furthermore, we have

$$\sum_{n=2}^{N} f(n) = L(P_0, f) \leq supL(P, f), \; for \; all \; P \; of \; [1, N]$$

Given that we have shown $\int_1^N f(x)dx$ exists and $\int_1^N f(x)dx \leq \int_1^\infty f(x)dx$ for any $N \geq 1$ above, we know that

$$\sum_{n=2}^{N} f(n) \leq \int_1^N f(x)dx \leq \int_1^\infty f(x)dx = l$$

Hence, $\sum_{n=2}^{N} f(n)$ is bounded above. Furthermore, the partial sum $\sum_{n=1}^{N} f(n) = f(1) + \sum_{n=2}^{N} f(n)$, and it is bounded above as well. Together with $f(n) \geq 0$, we know that $\sum_{n=1}^{\infty} f(n)$ converges.

Next, we prove that if $\sum_{n=1}^{\infty} f(n)$ converges, $\int_1^\infty f(x)dx$ converges.

Still use the same partition P_0 used above. Let $P_0 = (x_0 = 1, \cdots, x_{N-1} = N)$ (each $x_i = i + 1$). Because f monotonically decreases on $[1, \infty]$, $f(x) \leq f(n)$ for all $x \in [n, n+1]$. Thus, on each subinterval $[n, n+1]$ of P_0, we have

$$\int_n^{n+1} f(x)dx \leq \int_n^{n+1} f(n)dx = f(n)$$

Let $b_N = \int_1^N f(x)dx$, then

$$b_N = \sum_{n=1}^{N-1} \int_n^{n+1} f(x)dx \leq \sum_{n=1}^{N-1} f(n)$$

Because $f(x) \geq 0$ and $\sum_{n=1}^{\infty} f(n)$ converges, we further have

$$b_N \leq \sum_{n=1}^{N-1} f(n) \leq \sum_{n=1}^{\infty} f(n) < \infty$$

We have shown that b_N is bounded above for any natural number N. Next, we show that b_N is monotonically increasing.

$$b_{N+1} = \int_1^{N+1} f(x)dx = b_N + \int_N^{N+1} f(x)dx$$

Given that $f(x) \geq 0$, we have $\int_N^{N+1} f(x)dx \geq 0$. Thus, $b_{N+1} \geq b_N$. Because $\{b_N\}$ is bounded and monotonically increasing, we know that $\{b_N\}$ converges and hence $\int_1^\infty f(x)dx$ converges as well. $\square$

Extra problem.

See the attached.