26. <u>Lemma</u> f diff. on (a,b), cont on $[a,b]$, $f(a)=0$ and $|f'(x)| \le A\,|f(x)|$ for $x \in (a,b)$. Then $f(x)=0$ for $a \le x \le \min\{a+\frac{1}{A}, b\}$.

<u>Pf</u> Let $x_0 \in (a,b]$ s.t. $a \le x_0 < a+\frac{1}{A}$ (note strict $<$).

Let $M_0 = \sup_{x\in[a,x_0]} |f(x)|$, $M_1 = \sup_{x\in(a,x_0]} |f'(x)|$.

Pick $x_1 \in [a,x_0]$ s.t. $|f(x_1)| = M_0$ (using compactness and continuity.)

Suppose $M_0 > 0$. Then $x_1 > a$.

By MVT $\exists\, x_2 \in (a,x_1)$ s.t. $M_0 = |f(x_1) - f(a)| = |f'(x_2)(x_1-a)|$

$$\le A\,|f(x_2)|\cdot|x_1-a| < A\cdot M_0\cdot\frac{1}{A} = M_0.$$

The strict ineq $M_0 < M_0$ is a contradiction, so $M_0 = 0$.

This shows $f(x)=0$ for $x \in [0,x_0]$, for any $x_0 < \min\{a+\frac{1}{A}, b\}$.

By continuity, $f(x)=0$ up to $\min\{a+\frac{1}{A}, b\}$. $\square$

Now we complete the problem.

Let $a_0 = \sup\{t\in[a,b] : f(x)=0 \text{ for } x\in[a,t]\}$.

We know $a_0 \geq \min\{a+\frac{1}{A}, b\}$. If $a_0 = b$ we are done.

Suppose $a_0 < b$. Then by cont'y $f(a_0)=0$. Apply the lemma to f on $[a_0, b]$. Then lemma implies that $f=0$ on $[a_0, (a_0+\frac{1}{A})\wedge b]$. In other words, a_0 could not have been the supremum defined above.

This shows that a_0 must equal b.