4. $f, g : X \to Y$ cont., $E \subseteq X$ dense.

<u>Claim</u>: $f(E)$ is dense in $f(X)$.

Enough to show that, given $x \in X$, $\exists$ a sequence $x_n \in E$ s.t. $f(x_n) \longrightarrow f(x)$. But this is immediate: since $E$ is dense in $X$, $\exists$ sequence $x_n \in E$ s.t. $x_n \longrightarrow x$. (Constant sequence $x_n = x$ is a possibility here.) Then by continuity $f(x_n) \to f(x_n)$.

$\square$

<u>Claim</u> $f = g$ on $E$ implies $f = g$ on all of $X$.

<u>Pf</u>: Given $x \in X$, find sequence $\ni x_n \longrightarrow x$. Then by continuity of $f$ and $g$,

$$f(x) = \lim f(x_n) = \lim g(x_n) = g(x).$$

$\square$

11. $f: X \to Y$ uniformly cont.

Claim $\{x_n\}$ Cauchy in $X \implies \{f(x_n)\}$ Cauchy in $Y$.

Pf. Let $\varepsilon > 0$. Using uniform continuity, pick $\delta > 0$ s.t. $[x, y \in X$ and $d(x,y) < \delta] \implies d(f(x), f(y)) < \varepsilon$.

Using Cauchy prop of $\{x_n\}$, pick $N$ s.t.
$m, n \geq N \implies d(x_m, x_n) < \delta$.
Then $m, n \geq N \implies d(f(x_m), f(x_n)) < \varepsilon$, which shows $\{f(x_n)\}$ Cauchy. $\square$

$E \subseteq X$ dense, $f: E \to \mathbb{R}$ unif. cont.

Claim $\exists$ cont. $g: X \to \mathbb{R}$ s.t. $g(x) = f(x)$ for $x \in E$.

Pf. Step 1. Suppose $x_n, y_n \in E$ and $x_n \to x$, $y_n \to x$, for some $x \in X$. Then $\lim f(x_n) = \lim f(y_n)$ exists in $\mathbb{R}$.

Pf of Step 1. By the claim above, $\{f(x_n)\}$ and $\{f(y_n)\}$ are Cauchy sequences in $\mathbb{R}$. By the completeness of $\mathbb{R}$ they converge to some limits $f(x_n) \to s$ and $f(y_n) \to t$. We need to show that $s = t$.
Suppose $s \neq t$. Then $\exists \varepsilon > 0$ s.t. $|s - t| > \varepsilon$.
Using unif. cont'y of $f$, pick $\delta > 0$ s.t. $u, v \in E$ & $d(u,v) < \delta$ $\implies |f(u) - f(v)| < \varepsilon/4$. Then pick $N$ s.t.
$n \geq N \implies d(x_n, x) < \delta/2$ and $d(y_n, x) < \delta/2$. and
$|f(x_n) - s| < \varepsilon/8$ and $|f(y_n) - t| < \varepsilon/8$.

Now put these together:

$n \geq N \implies d(x_n, y_n) \leq d(x_n, x) + d(x, y_n) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$

so $n \geq N \implies |f(x_n) - f(y_n)| < \varepsilon/4.$

Consequently, still with $n \geq N$,

$$|s - t| \leq |s - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - t|$$
$$< \varepsilon/8 + \varepsilon/4 + \varepsilon/8 = \varepsilon/2,$$

contradicting $|s - t| > \varepsilon$.

We have shown that $s = t$. $\square$

Now given $x \in X$ we define $g(x) = \lim_{n\to\infty} f(x_n)$ for any sequence

Step 1 shows that this def. cannot give two distinct values. The assumption that $E$ is dense implies that every $x \in X$ has at least one such sequence. (Note: if $x \in E$ then the constant sequence $x_n = x$ is admissible now.)

Thus $g$ is well-defined. It remains to check its continuity. Let $\varepsilon > 0$. Using u.c. of $f$ on $E$, pick $\delta > 0$ s.t. $u, v \in E$ and $d(u,v) < \delta \implies |f(u) - f(v)| < \frac{\varepsilon}{4}$.

Let $x, y \in X$ satisfy $d(x,y) < \delta/4$.

Pick sequences $E \ni x_n \to x$, $E \ni y_n \to y$.

Then by def. $f(x_n) \to g(x)$ and $f(y_n) \to g(y)$.

Pick $n$ large enough so that $d(x_n, x) < \delta/4$, $d(y_n, y) < \delta/4$, $|f(x_n) - g(x)| < \varepsilon/4$ and $|f(y_n) - g(y)| < \varepsilon/4$.

Then $d(x_n, y_n) \le d(x_n, x) + d(x,y) + d(y, y_n) < \delta$

$\Longrightarrow \quad |f(x_n) - f(y_n)| < \varepsilon/4.$

And next $|g(x) - g(y)| \le |g(x) - f(x_n)|$

$+ |f(x_n) - f(y_n)| + |f(y_n) - g(y)| < \varepsilon.$

We have shown: this $\delta > 0$ satisfies

$d(x,y) < \delta/4 \Longrightarrow |g(x) - g(y)| < \varepsilon.$

Thus $g$ is unif. cont.

<u>Example</u> of $f$ continuous on $E$ but not unif. cont. and such that extension to $X$ is not possible:

$X = [0,1], \quad E = [0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1]$

$$f(x) = \begin{cases} 0, & x \in [0, \tfrac{1}{2}) \\ 1, & x \in (\tfrac{1}{2}, 1] \end{cases}$$

$f$ is continuous on $E$ because given $x \in E$, $f$ is constant in a nbhd $(x-\delta, x+\delta)$ of $x$ that lies inside $E$.

But $f$ cannot be extended continuously to $x = \tfrac{1}{2}$ because $f(\tfrac{1}{2}-) = 0 \neq 1 = f(\tfrac{1}{2}+)$.