HW 6 SOLUTION SKETCHES

5. To show $\overline{\lim}\,(a_n+b_n) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n$. (1)

<u>SOL-1</u>

Pick subseq $\{n_k\}$ s.t. $a_{n_k}+b_{n_k} \longrightarrow \overline{\lim}\,(a_n+b_n)$.
The set of subsequential limits of $\{a_{n_k}\}_{k\in\mathbb{N}}$ is not empty. Hence $\exists\, a \in [-\infty,\infty]$ and subseq $a_{n_{k_j}} \longrightarrow a$ as $j \to \infty$. Apply same reasoning to $\{b_{n_{k_j}}\}_{j\in\mathbb{N}}$ to find a further subseq with a limit: $b_{n_{k_{j_\ell}}} \longrightarrow b$ as $\ell \to \infty$. Now

$$\overline{\lim}\,(a_n+b_n) = \lim_{\ell\to\infty}\left(a_{n_{k_{j_\ell}}} + b_{n_{k_{j_\ell}}}\right) = a+b$$

$$\leq \overline{\lim}\, a_n + \overline{\lim}\, b_n.$$

<u>SOL 2</u>

We can assume $\overline{\lim}\, a_n + \overline{\lim}\, b_n < \infty$, otherwise nothing to prove. Let $\varepsilon>0$. Then

$\exists\, N_1$ s.t. $n \geq N_1 \implies a_n \leq \overline{\lim}\, a_n + \varepsilon/2$

$\exists\, N_2$ s.t. $n \geq N_2 \implies b_n \leq \overline{\lim}\, b_n + \varepsilon/2$.

Thus $n \geq N_1 \vee N_2 \implies a_n + b_n \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n + \varepsilon$.

This implies $\overline{\lim}\,(a_n+b_n) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n + \varepsilon$.

Since this holds $\forall\, \varepsilon>0$, we have (1).

6 (a) $a_n = \sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$. Diverges, compare with $\dfrac{1}{3\sqrt{n}}$.

(b) $a_n = \dfrac{\sqrt{n+1}-\sqrt{n}}{n} = \dfrac{1}{n\,(\sqrt{n+1}+\sqrt{n})}$. Converges, compare with $\dfrac{1}{n^{3/2}}$.

7. Let $A = \sum a_n$, $B = \sum \frac{1}{n^2}$, both known to be finite. For any $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \sqrt{a_n} \cdot \frac{1}{n} \le \left(\sum_{n=1}^{N} a_n\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} \frac{1}{n^2}\right)^{\frac{1}{2}} \le \sqrt{AB}.$$

Nonnegative series $\Rightarrow$ bdd partial sums imply convergence.

14 (c) Let $a_n = 2^{-n}$, $b_n = \begin{cases} k & \text{if } n = 2^k;\ k=1,2,3,\dots \\ 0 & \text{otherwise} \end{cases}$

and $s_n = a_n + b_n$.

$$\sum_{n=1}^{2^m} b_n = \sum_{k=1}^{m} k = \frac{m(m+1)}{2} \le m^2.$$

You should see that $\sigma_n \to 0$, $\overline{\lim}\, s_n = \infty$.

(d) $$\sum_{k=1}^{n} k a_k = \sum_{k=1}^{n} k s_k - \sum_{k=1}^{n} k s_{k-1} = \sum_{k=1}^{n} k s_k - \sum_{k=0}^{n-1} (k+1) s_k = n s_n - \sum_{k=0}^{n-1} s_k$$

$$= (n+1) s_n - \sum_{k=0}^{n} s_n.$$

For the limit just use (a) for the average $\frac{1}{n+1} \sum_{k=1}^{n} k a_k$.

16. (a) Step 1 $x_n > \sqrt{\alpha}$ $\forall n$.

Inductive proof: True for x_1 by assumption.

Suppose $x_n > \sqrt{\alpha}$. Then

$$x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(x_n - 2\sqrt{\alpha} + \frac{\alpha}{x_n}\right) = \frac{1}{2}\left(\sqrt{x_n} - \sqrt{\frac{\alpha}{x_n}}\right)^2 > 0.$$

Step 2 $\{x_n\}$ decreases monotonically.

$$x_n > \sqrt{\alpha} \Rightarrow x_n^2 > \alpha \Rightarrow \frac{1}{2}x_n > \frac{1}{2}\frac{\alpha}{x_n}$$

$$\Rightarrow x_n > \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \Rightarrow x_n > x_{n+1}.$$

Step 3 $\{x_n\}$ is bounded below (step 1), hence a limit $x_n \to c$ exists.

$$\text{Thm 3.3} \Rightarrow \quad x_{n+1} - \frac{x_n}{2} + \frac{\alpha}{2x_n} \to c - \frac{c}{2} + \frac{\alpha}{2c} = \frac{c}{2} - \frac{\alpha}{2c}.$$

But by construction $x_{n+1} - \frac{x_n}{2} + \frac{\alpha}{2x_n} = 0$ for all n, and so the limit is also 0. From this,

$$\frac{c}{2} - \frac{\alpha}{2c} = 0 \Rightarrow c = \frac{\alpha}{c} \Rightarrow c^2 = \alpha \Rightarrow c = \sqrt{\alpha}.$$

(Since $x_n > \sqrt{\alpha}$, the limit c is positive, so it cannot be $-\sqrt{\alpha}$.)