**521 Analysis I Spring 2011 Homework 5**
Due Monday February 28

**1.** Part (a) is a small but extremely useful observation. It comes in handy in situations where you want $x \le a$ but the convenient thing to show is $x \le a + \varepsilon$ for all $\varepsilon > 0$.

(a) Let $a, x \in [-\infty, \infty)$. Assume that for each number $c$ such that $c > a$, we have the inequality $x \le c$. Show that then $x \le a$.

(b) Let $\{s_n\}$ be a real sequence with a limit $s_n \to s$. Suppose $s_n \le a$ for each $n$. Use the definition of the limit and part (a) to show that $s \le a$.

**2.** Let $\{s_n\}$ be a real sequence. The purpose of this exercise is to derive this formula:

$$\overline{\lim_{n\to\infty}}\, s_n = \inf_{\ell\in\mathbb{N}}\Big( \sup_{n:n\ge\ell} s_n \Big). \qquad (1)$$

Equation (1) is often given as the definition of limsup.

First let's make sure we understand what is claimed. Let $s = \overline{\lim}\, s_n$ be as defined in Rudin's text, namely $s = \sup E$ where

$$E = \{x \in [-\infty, \infty] : \exists \text{ subsequence } s_{n_k} \to x \}.$$

Let $t$ denote the right-hand side of (1). Then the formula on the right of (1) means that $t = \inf\{y_\ell : \ell \in \mathbb{N}\}$ where $y_\ell = \sup\{s_n : n \ge \ell\}$.

With these definitions of $s$ and $t$, the task is to show $s = t$. Below is an outline you can follow, but you are also welcome to find your own way.

*Step 1:* $t \le s$. First dispose of the case $s = \infty$. If $s < \infty$, take $c > s$ and try to show that $t \le c$. Make use of Thm 3.17(b) and that $t \le y_\ell$ for each $\ell$.

*Step 2:* $t \ge s$. Show that each $y_\ell$ is an upper bound of the set $E$. From this you can argue that $s \le y_\ell$ for each $\ell$.