Ch 3

14 (a) Let $\varepsilon > 0$. We need to show the existence of $M \in \mathbb{N}$ s.t. $n \geq M \Rightarrow |\sigma_n - s| \leq \varepsilon$.

Let's do a preliminary calculation, with $0 < N < n$.

$$\sigma_n - s = \frac{1}{n+1} \sum_{i=0}^{N-1} (s_i - s) + \frac{1}{n+1} \sum_{i=N}^{n} (s_i - s)$$

so by the triangle inequality

$$|\sigma_n - s| \leq \left| \frac{1}{n+1} \sum_{i=0}^{N-1} (s_i - s) \right| + \frac{1}{n+1} \left| \sum_{i=N}^{n} (s_i - s) \right|$$

$$\leq \frac{1}{n+1} \sum_{i=0}^{N-1} |s_i - s| + \frac{1}{n+1} \sum_{i=N}^{n} |s_i - s|.$$

This tells us how to make our choices.
Choose N so that $n \geq N \Rightarrow |s_i - s| \leq \frac{\varepsilon}{2}$.
Given N, choose $M \geq N$ so that $\frac{1}{M} \sum_{i=0}^{N-1} |s_i - s| \leq \frac{\varepsilon}{2}$.

Now, for $n \geq M$

$$|\sigma_n - s| \leq \frac{M}{n+1} \cdot \underbrace{\frac{1}{M} \sum_{i=0}^{N-1} |s_i - s|}_{\leq \varepsilon/2} + \frac{1}{n+1} \sum_{i=N}^{n} \underbrace{|s_i - s|}_{\leq \varepsilon/2}$$

$$\leq \frac{M}{n+1} \cdot \frac{\varepsilon}{2} + \frac{n-N+1}{n+1} \cdot \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$$

Ch 3

20. Given $\varepsilon > 0$. Need to show the existence of $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow d(p_n, p) \leq \varepsilon$.

(1) By the Cauchy assumption we can choose N_0 s.t. $m, n \geq N_0 \Longrightarrow d(p_m, p_n) \leq \varepsilon/2$.

(2) By the assumption $p_{n_i} \longrightarrow p$ we can choose I s.t. $i \geq I \Rightarrow d(p_{n_i}, p) \leq \varepsilon/2$.

Now let $N = \max\{N_0, n_I\}$ (the maximum of the two numbers).

Let $n \geq N$. Pick $i \geq I$ large enough so that also $n_i \geq N_0$. Then by the triangle inequality

$$d(p_n, p) \leq \underbrace{d(p_n, p_{n_i})}_{\leq \frac{\varepsilon}{2} \text{ by (1)}} + \underbrace{d(p_{n_i}, p)}_{\leq \frac{\varepsilon}{2} \text{ by (2)}} \leq \varepsilon.$$

From p 51. Suppose $p_n \to p$. Let $\{p_{n_i}\}$ be a subsequence. Let $\varepsilon > 0$. Pick N s.t. $n \geq N \Longrightarrow d(p_n, p) \leq \varepsilon$. Pick I s.t. $n_I \geq N$.

Then $i \geq I \Longrightarrow n_i \geq n_I \geq N \Longrightarrow d(p_{n_i}, p) \leq \varepsilon$.
This says $p_{n_i} \longrightarrow p$ as $i \to \infty$.