c. If $G \subseteq E$ and G open, then $G \subseteq E^\circ$.

Proof. Let G be an open set such that $G \subseteq E$. Take point $p \in G$. By definition of an open set, p is an interior point of G, meaning there exists neighborhood $N_r(p) \subseteq G \subseteq E$. Thus p is an interior point of E. Thus, $p \in E^\circ$. Therefore $G \subseteq E^\circ$. □

d. The complement of E° is the closure of the complement of E.

Proof. ($\subseteq$) Take the point $x \notin E^\circ$. Then, every neighborhood around x is not a subset of E. Thus, every neighborhood around x contains a point not in E. Therefore, x is a limit point of E^c.

Thus, the complement of E° is a subset of the closure of the complement of E. □

Proof. ($\supseteq$) Note that $E^\circ \subseteq E$. Consider point x not in E, then x not in E°. So $(E^\circ)^c \supseteq E^c$.

The complement of an open set is always closed, thus by part a, $(E^\circ)^c$ is closed. By Theorem 2.27, if a set is the subset of a closed set, then its closure is also a subset. So the closure of E^c is a subset of $(E^\circ)^c$.

Therefore, since the complement of E° is a subset and a superset of the closure of the complement of E, the complement of E° is the closure of the complement of E. □

e. Do E and $\bar{E}$ always have the same interiors? No. Consider the set on $\mathbb{R}$, $E = (0,1) \cup (1,2)$. Then $E^\circ = E$, but $\bar{E} = [0,2]$ and its interior is the set $(0,2)$.

f. Do E and E° always have the same closures? No. Consider the set on $\mathbb{R}$, $E = (0,1) \cup \{2\}$. Then, $E^\circ = (0,1)$. The closure of E is the set $[0,1] \cup \{2\}$, but the closure of E° is the set $[0,1]$.

2.22 Separable $\mathbb{R}^k$

coordinate direction

The set $\mathbb{R}^k$ is separable. That is, it has a dense, countable subset.

Proof. Consider the set $\mathbb{Q}^k \subset \mathbb{R}^k$. By theorem 2.13, the set $\mathbb{Q}^k$ is countable.

Let $p \in \mathbb{R}^k$. If $p \notin \mathbb{Q}^k$, then consider a neighborhood of radius r around $p = (p_1, p_2, \ldots, p_k)$, denoted $N_r(p)$. By theorem 1.20, for every ~~dimension~~ i we can find a rational number q_i such

$\frac{r}{\sqrt{k}}$ that $p_i < q_i < p_i + r$. Thus there exists a point $q \in N_r(p)$ such that q also in $\mathbb{Q}^k$. Therefore, since this was an arbitrary neighborhood, p is a limit point of $\mathbb{Q}^k$.

Therefore, $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$, and $\mathbb{R}^k$ is separable. □

2.29

Every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments.

Proof. Let $G \subset \mathbb{R}$, where G is an open set.

Construction of "largest open interval around a rational". Let $q \in \mathbb{Q}$ and $q \in G$. Define a set $V = \{x : [x,q] \subseteq G \text{ or } [q,x] \subseteq G\}$. Consider the open interval around q, (a_q, b_q), where $a_q = \inf V, b_q = \sup V$, called the largest open interval around q.

Intervals are Disjoint. Let $p, q \in G$ and $p, q \in \mathbb{Q}$. Without loss of generality, assume $p < q$. Let V_p, V_q be the largest open intervals around p, q, respectively. We then consider the two possible cases,

Case: $[p,q] \subseteq G$. Because $[p,q] \subseteq G$, and since $p < q$, the statement for $x \in G$ that $[x,q] \subseteq G$ is equivalent to the statement that $[x,p] \subseteq G$. This implies that $\inf V_p = \inf V_q$.

Because $[p,q] \subseteq G$, and since $p < q$, the statement for $x \in G$ that $[p,x] \subseteq G$ is equivalent to the statement that $[q,x] \subseteq G$. This implies that $\sup V_p = \sup V_q$.

Therefore, the two open intervals are the same.