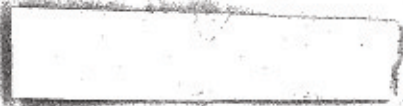


MATH 521: ASSIGNMENT 3



2.7 CLOSURES OF UNIONS

Let A_1, A_2, A_3, \dots be subsets of a metric space.

a. If $B_n = \bigcup_{i=1}^n A_i$, then $\bar{B}_n = \bigcup_{i=1}^n \bar{A}_i$ for $n \in \mathbb{N}$.

Let $B_n = \bigcup_{i=1}^n A_i$. Let $C_n = \bigcup_{i=1}^n \bar{A}_i = B_n \cup \bigcup_{i=1}^n A'_i$, where A'_i indicates the set of limit points of A_i .

Proof. (\subseteq) Since C_n the union of a finite number of closed sets, Theorem 2.24 implies that C_n is also closed. Since $B_n \subseteq C_n$ and C_n closed, by theorem 2.27, $\bar{B}_n \subseteq C_n$. \square

Proof. (\supseteq) Take $x \in C_n$. If x in some A_k , then obviously $x \in B_n \implies x \in \bar{B}_n$.

The only other case is that x is a limit point of some A_k . i.e. $x \in A'_k$. Then, every neighborhood around x contains some other point in A_k . Since $A_k \subseteq B_n$, then every neighborhood around x contains some other point in B_n . Thus, x is a limit point of B_n and $x \in \bar{B}_n$.

Therefore, $\bar{B}_n \subseteq \bigcup_{i=1}^n \bar{A}_i$ and $\bar{B}_n \supseteq \bigcup_{i=1}^n \bar{A}_i$, so $\bar{B}_n = \bigcup_{i=1}^n \bar{A}_i$. \square

b. If $B = \bigcup_{i=1}^{\infty} A_i$, then $\bar{B} \supseteq \bigcup_{i=1}^{\infty} \bar{A}_i$.

Proof. Let $B = \bigcup_{i=1}^{\infty} A_i$. Take $x \in \bigcup_{i=1}^{\infty} \bar{A}_i$. If x in some A_k , then obviously $x \in B$. Which implies $x \in \bar{B}$.

The only other case is that x is a limit point of some A_k . i.e. $x \in A'_k$. Then, every neighborhood around x contains some other point in A_k . Since $A_k \subseteq B$, then every neighborhood around x contains some other point in B . Thus, x is a limit point of B and $x \in \bar{B}$.

Therefore, $\bar{B} \supseteq \bigcup_{i=1}^{\infty} \bar{A}_i$. \square

Example. Show that this inclusion can be proper.

Take $A_i = \{1/i\}$. Then (as shown in class), $\bar{B} = \{1/i : i \in \mathbb{N}\} \cup \{0\}$. This is a proper super set of $\bigcup_{i=1}^{\infty} \bar{A}_i = \{1/i : i \in \mathbb{N}\}$.

2.9 INTERIOR POINTS

Let E° denote the set of interior points of set E . Use metric $d(x, y)$.

a. E° is always open.

Proof. Let $p \in E^\circ$. Since p is an interior point of E , there exists a neighborhood $N_r(p) \subseteq E$. Take $q \in N_r(p)$. Since $d(p, q) < r$, there exists a neighborhood of radius $r' = r - d(p, q)$ such that $N_{r'}(q) \subseteq N_r(p) \subseteq E$. This means ~~that~~ that every point $q \in N_r(p)$ is an interior point of E . Thus, $N_r(p) \subseteq E^\circ$. Therefore, every point $p \in E^\circ$ is an interior point of E° . Thus, E° is open. \square

b. E is open $\iff E = E^\circ$.

Proof. \Leftarrow By part a, E° is open. Thus, if $E = E^\circ$, then E is open. \square

Proof. \Rightarrow If E is open, then all of its points are interior points. Then, E° , the set of all interior points of E , equals E . \square

Date: Due Monday, February 7th.