MATH 521: ASSIGNMENT 3

2.7 Closures of Unions

Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

a. If $B_n = \bigcup_{i=1}^n A_i$, then $\bar{B}_n = \bigcup_{i=1}^n \bar{A}_i$ for $n \in \mathbb{N}$.

Let $B_n = \bigcup_{i=1}^n A_i$. Let $C_n = \bigcup_{i=1}^n \bar{A}_i = B_n \cup \bigcup_{i=1}^n A_i'$, where A_i' indicates the set of limit points of A_i.

Proof. ($\subseteq$) Since C_n the union of a finite number of closed sets, Theorem 2.24 implies that C_n is also closed. Since $B_n \subseteq C_n$ and C_n closed, by theorem 2.27, $\bar{B}_n \subseteq C_n$. □

Proof. ($\supseteq$) Take $x \in C_n$. If x in some A_k, then obviously $x \in B_n \implies x \in \bar{B}_n$.

The only other case is that x is a limit point of some A_k. i.e. $x \in A_k'$. Then, every neighborhood around x contains some other point in A_k. Since $A_k \subseteq B_n$, then every neighborhood around x contains some other point in B_n. Thus, x is a limit point of B_n and $x \in \bar{B}_n$.

Therefore, $\overline{B_n} \subseteq \bigcup_{i=1}^n \bar{A}_i$ and $\overline{B_n} \supseteq \bigcup_{i=1}^n \bar{A}_i$, so $\overline{B_n} = \bigcup_{i=1}^n \bar{A}_i$ □

b. If $B = \bigcup_{i=1}^\infty A_i$, then $\bar{B} \supseteq \bigcup_{i=1}^\infty \bar{A}_i$.

Proof. Let $B = \bigcup_{i=1}^\infty A_i$. Take $x \in \bigcup_{i=1}^\infty \bar{A}_i$. If x in some A_k, then obviously $x \in B$. Which implies $x \in \bar{B}$.

The only other case is that x is a limit point of some A_k. i.e. $x \in A_k'$. Then, every neighborhood around x contains some other point in A_k. Since $A_k \subseteq B$, then every neighborhood around x contains some other point in B. Thus, x is a limit point of B and $x \in \bar{B}$.

Therefore, $\overline{B} \supseteq \bigcup_{i=1}^\infty \bar{A}_i$. □

Example. Show that this inclusion can be proper.

Take $A_i = \{1/i\}$. Then (as shown in class), $\bar{B} = \{1/i : i \in \mathbb{N}\} \cup \{0\}$. This is a proper super set of $\bigcup_{i=1}^\infty \bar{A}_i = \{1/i : i \in \mathbb{N}\}$.

2.9 Interior Points

Let E° denote the set of interior points of set E. Use metric $d(x, y)$.

a. E° is always open.

Proof. Let $p \in E^\circ$. Since p is an interior point of E, there exists a neighborhood $N_r(p) \subseteq E$. Take $q \in N_r(p)$. Since $d(p,q) < r$, there exists a neighborhood of radius $r' = r - d(p,q)$ such that $N_{r'}(q) \subseteq N_r(p) \subseteq E$. This means [illegible] that every point $q \in N_r(p)$ is an interior point of E. Thus, $N_r(p) \subseteq E^\circ$. Therefore, every point $p \in E^\circ$ is an interior point of E°. Thus, E° is open. □

b. E is open $\iff E = E^\circ$.

Proof. $\Leftarrow$ By part a, E° is open. Thus, if $E = E^\circ$, then E is open. □

Proof. $\Rightarrow$ If E is open, then all of its points are interior points. Then, E°, the set of all interior points of E, equals E. □

Date: Due Monday, February 7th.