$\boxed{2.5}$ Construct a bounded set in $\mathbb{R}$ with exactly three limit points

I wish to prove that the set $S = \{x \in \mathbb{R} : 1/n, n \in \mathbb{N}\} \cup \{x \in \mathbb{R} : 1/n + 1, n \in \mathbb{N}\} \cup \{x \in \mathbb{R} : 1/n + 2, n \in \mathbb{N}\}$ is bounded and has exactly three limit points.

Consider the neighborhood of radius 3 around the point 1, denoted $N_3(1)$. Since $1/n > 0, \forall n \in \mathbb{N}$, then $s > 0, \forall s \in S$. Also, since $1/n \leq 1, \forall n \in \mathbb{N}$, then $s \leq 2, \forall s \in S$. Therefore, $S \subset N_3(1)$, meaning S is bounded.

Next, consider the point 0. For any radius r, we can find a point in S by taking n such that $1/n < r$. Next consider the point 1. For any radius r, we can find a point in S by taking n such that $1/n + 1 < r + 1$. Similarly, around the point 2, for any radius r we can find a point in S by taking n such that $1/n + 2 < r + 2$. Therefore the points 0, 1, and 2 are limit points.

Now, take $s \in S - \{0, 1, 2\}$. Then, s is of the form $s = 1/n + z, n \in \mathbb{N}, z \in \{0, 1, 2\}$. This means that for any s, there are no other points in S in the neighborhood $N_{1/(n+1)}(s)$. Therefore, s is not a limit point.

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Thus, the set S is bounded with exactly 3 limit points.

$\boxed{2.11}$ Metrics

For $x, y \in \mathbb{R}$, determine if the following functions are metrics.

(1)

$$d_1(x, y) = (x - y)^2$$

Checking the triangle inequality property, pick some arbitrary $z \in \mathbb{R}$. Then,

$$\begin{aligned} d_1(x, z) &= (x - z)^2 \\ &= x^2 - 2xz + z^2 \end{aligned}$$

and

$$\begin{aligned} d_1(x, y) + d_1(y, z) &= (x - y)^2 + (y - z)^2 \\ &= x^2 - 2xy + y^2 + y^2 - 2yz + z^2 \\ &= x^2 - 2xy + 2y^2 - 2yz + z^2 \\ (d_1(x, y) + d_1(y, z)) - d_1(x, z) &= x^2 - 2xy + 2y^2 - 2yz + z^2 - x^2 + 2xz - z^2 \\ &= 2xz - 2xy - 2yz + 2y^2 \end{aligned}$$

Thus, picking $z = 0$ and $0 < y < x$ breaks the triangle inequality property. Therefore, d_1 is not a metric.

(2)

$$d_2(x, y) = \sqrt{|x - y|}$$

$\sqrt{z} = 0$ if and only if $z = 0$, and $|x - y| = 0$ if and only if $x = y$. Thus, $d_2(x, y) = 0$ if and only if $x = y$. Also, for $z > 0$, $\sqrt{z} > 0$. Thus, $d_2(x, y) > 0$ if $x \neq y$.

Since $|x - y| = |y - x|$, then $\sqrt{|x - y|} = \sqrt{|x - y|}$.

Finally, we will show that the triangle inequality holds for d_2. Let $z \in \mathbb{R}$. Note that the first statement is true by Thm 1.37 in Rudin.

$$\begin{aligned} |x - y| + |y - z| &\geq |x - z| \\ |x - y| + 2\sqrt{|x - y|}\sqrt{|y - z|} + |y - z| &\geq |x - z| \\ (\sqrt{|x - y|} + \sqrt{|y - z|})^2 &\geq |x - z| \\ \sqrt{|x - y|} + \sqrt{|y - z|} &\geq \sqrt{|x - z|} \\ d_2(x, y) + d_2(y, z) &\geq d_2(x, z) \end{aligned}$$

Therefore, all the properties of a metric are held, and d_2 is a metric.

(3)

$$d_3(x, y) = |x^2 - y^2|$$

No.
For $x = 1, y = -1$

$$d_3(1,-1) = |x^2 - y^2| = |(1)^2 - (-1)^2| = 0$$

Which violates the property $d(x,y) \neq 0$ for $x \neq y$.

(d_4)

No.
For $x = 1$

$$d_4(x,x) = |x - 2x| = 1$$

Which violates the property $d(x,x) = 0$.

(d_5)

Yes.
(1)

$$d_5(x,x) = \frac{|x-x|}{1+|x-x|} = \frac{0}{1+0} = 0.$$

(2) By $|x-y| = |y-x|$

$$d_5(x,y) = \frac{|x-y|}{1+|x-y|} = \frac{|y-x|}{1+|y-x|} = d_5(y,x).$$

(3)

$$\frac{p}{1+p} \leq \frac{q}{1+q} + \frac{r}{1+r}$$
$$\Leftrightarrow p(1+q)(1+r) \leq q(1+r)(1+p) + r(1+p)(1+q)$$
$$\Leftrightarrow p + pq + pr + pqr \leq (q + pq + qr + pqr) + (r + pr + qr + pqr)$$
$$\Leftrightarrow p \leq q + r + 2qr + pqr$$
$$\Leftrightarrow p \leq q + r$$

great !

Because $|x-z| \leq |x-y| + |y-z|$, it follows that $d_5(x,y)$ satisfies (3) and is, therefore, a metric.