MATH 521 - HOMEWORK 2

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Problem (1.15)**.** For $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^k$, the Schwarz inequality is given by

$$|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|,$$

We will show equality holds if and only if $\mathbf{x} = \alpha\mathbf{y}$ or $\mathbf{y} = \alpha\mathbf{x}$ for some $\alpha \in \mathbb{R}$.

Proof. Recall that the Schwarz inequality may be written as

$$\left(\sum x_i y_i\right)^2 \leq \left(\sum x_i^2\right)\left(\sum y_i^2\right).$$

For notational ease we denote

$$A = \sum x_i^2, \quad B = \sum y_i^2, \quad C = \sum x_i y_i.$$

Then equality is equivalent to

$$\begin{aligned} 0 &= AB - C^2 \\ &= B(AB - C^2) \\ &= B^2A - BC^2 \\ &= B^2 - 2BC^2 + C^2B \\ &= \sum \left(B^2x_i^2 - 2BCx_iy_i + C^2y_i^2\right) \\ &= \sum \left(Bx_i - Cy_i\right)^2 \end{aligned}$$

This implies

$$0 = \sum (Bx_i - Cy_i)^2.$$

Then $Bx_i = Cy_i$ for all i. Now we have two cases to consider.

Case I: $B = 0$. Then $\mathbf{y} = \mathbf{0}$, and we may write $\mathbf{y} = 0 \cdot \mathbf{x}$.

Case II: $B \neq 0$. Then $\mathbf{y} \neq \mathbf{0}$, so we may write $\mathbf{x} = \frac{C}{B}\mathbf{y}$.

Thus we have shown equality implies there exists a constant $\alpha \in \mathbb{R}$ such that either $\mathbf{x} = \alpha\mathbf{y}$ or $\mathbf{y} = \alpha\mathbf{x}$.

Now we show the reverse implication. We show $\mathbf{x} = \alpha\mathbf{y}$ implies equality.

$$\begin{aligned} |\mathbf{x} \cdot \mathbf{y}| &= |\alpha\mathbf{y} \cdot \mathbf{y}| \\ &= |\alpha||\mathbf{y} \cdot \mathbf{y}| \\ &= |\alpha||\mathbf{y}|^2 \\ &= |\mathbf{x}||\mathbf{y}| \end{aligned}$$

The same computation yields the case $\mathbf{y} = \alpha\mathbf{x}$.

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