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16/16 3

$\boxed{1.2}$ Irrational Square Root

Prove there is no rational number whose square is 12.
Assume that there exists a rational number x such that $x^2 = 12$. Since x is rational, it can be written m/n, where $gcd(m,n) = 1$.

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$$\begin{aligned} x^2 &= 12 \\ \frac{m^2}{n^2} &= 12 \\ m^2 &= 12n^2 \\ m^2 &= 3 \cdot 2^2 n^2 \end{aligned}$$

Thus m must have a prime factor of 3, $m = 3k$ for some integer k.

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$$\begin{aligned} (3k)^2 &= 3(2n)^2 \\ 3^2k^2 &= 3 \cdot 4n^2 \\ 3k^2 &= 4n^2 \end{aligned}$$

Thus n must have a prime factor 3, so $gcd(m,n) \geq 3$. This is a contradiction. Therefore, there is no rational number whose square is 12.

$\boxed{1.3}$ Field Multiplication Properties (Prop 1.15)

(a) If $x \neq 0$ and $xy = xz$ then $y = z$.
Assume $x \neq 0$ and $xy = xz$, the axioms (M on p.5) give

$$\begin{aligned} y = 1 \cdot y &= \frac{x}{x}y = \frac{xy}{x} \\ &= \frac{xz}{x} = \frac{x}{x}z \\ &= z \end{aligned}$$

(b) If $x \neq 0$ and $xy = x$ then $y = 1$.

$$\begin{aligned} xy &= x \\ xy &= x \cdot 1 \end{aligned}$$

By (a) $y = 1$.

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.

$$\begin{aligned} xy &= 1 \\ xy &= x\frac{1}{x} \end{aligned}$$

By (a) $y = 1/x$.

(d) If $x \neq 0$ then $1/(1/x) = x$.

Let $x = 1/z$, since $x \neq 0$, then $z \neq 0$. Also, let $y = \frac{1}{x}$. By (c)

$$\begin{aligned} xy &= 1 \\ x\frac{1}{x} &= 1 \\ \frac{1}{z}\frac{1}{1/z} &= 1 \\ z\frac{1}{z}\frac{1}{1/z} &= z \cdot 1 \\ \frac{1}{1/z} &= z \end{aligned}$$

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4

$\boxed{1.5}$

Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A)$$

Let $\inf A = \alpha$. Then by definition,

$$\forall \gamma \in A, \gamma \geq \alpha$$

Now, take the negative of both sides of the inequality.

$$\begin{aligned} \forall \gamma \in A, -\gamma \leq -\alpha \\ \forall \gamma' \in -A, \gamma' \leq -\alpha \end{aligned}$$

Thus $-\alpha$ is an upper bound of $-A$. Let $\beta \in \mathbb{R}, \beta < -\alpha$, then $-\beta > \alpha = \inf A$. Therefore, there exists some $\gamma \in A$ s.t. $\gamma < -\beta$. This means $-\gamma > \beta$. Since $-\gamma \in -A$, then β is not an upper bound of $-A$.

Therefore, $-\alpha$ is the supremum of $-A$, and

3|3

$$\begin{aligned} \inf A = \alpha &= -(-\alpha) \\ &= -\sup(-A) \end{aligned}$$

$\boxed{1.6}$ Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}$$

First, I'll prove some needed properties of exponentials.
Let x, y be integers and $y > 0$.

$$b^{xy} = (b^x)^y$$

This is trivially true by the definition of integer exponentiation.

$$(b^y)^{1/y} = b$$

is trivially true by definition of $b^{1/x} = y$ s.t. $y^x = b$ (Thm 1.21).

Also, for roots let $a, b, n \in \mathbb{Z}^+$ where $n = ab$. Let $x, y \in \mathbb{R}$ s.t. $y = x^{1/n}$. This is also written $y^n = x$.

$$\begin{aligned} x = y^n &= y^{ab} \\ &= (y^a)^b \\ x^{1/b} &= y^a \\ (x^{1/b})^{1/a} &= y = x^{\frac{1}{ab}} \end{aligned}$$

Without loss of generality, also $y = x^{\frac{1}{ab}} = (x^{1/a})^{1/b}$.

Let $r = x/y$, such that $gcd(x, y) = 1$. Let $d = gcd(m, n)$ and $f = gcd(p, q)$. Thus, $m = dx, n = dy, p = fx, q = fy$.

Then, we simplify $(b^m)^{1/n}$,

$$\begin{aligned} (b^m)^{\frac{1}{n}} &= (b^{dx})^{\frac{1}{dy}} \\ &= ((b^x)^d)^{\frac{1}{dy}} \\ &= (((b^x)^d)^{\frac{1}{d}})^{\frac{1}{y}} \\ &= (b^x)^{\frac{1}{y}} \end{aligned}$$

Similarly, simplifying $(b^p)^{1/q}$,

$$\begin{aligned} (b^p)^{\frac{1}{q}} &= (b^{fx})^{\frac{1}{fy}} \\ &= ((b^x)^f)^{\frac{1}{fy}} \\ &= (((b^x)^f)^{\frac{1}{f}})^{\frac{1}{y}} \\ &= (b^x)^{\frac{1}{y}} \end{aligned}$$

Therefore,

$$(b^m)^{1/n} = (b^p)^{1/q}$$

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

Since r, s are rational, they can be written $r = m/n$ and $s = p/q$ where $m, n, p, q \in \mathbb{Z}$ and $n > 0, q > 0$.

$$\begin{aligned} b^{r+s} &= b^{\frac{m}{n}+\frac{p}{q}} = b^{\frac{mq+pn}{nq}} \\ &= (b^{mq+pn})^{\frac{1}{nq}} \\ &= (b^{mq}b^{pn})^{\frac{1}{nq}} \text{ by multiplicative commutativity} \\ &= b^{\frac{mq}{pn}} b^{\frac{pn}{nq}} \text{ by corollary to Thm 1.21} \\ &= b^{\frac{m}{n}} b^{\frac{p}{q}} = b^r b^s \end{aligned}$$

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational.

If $r, s \in \mathbb{Q}$ and $s < r$,

$$\begin{aligned} b^s - b^r &= b^s - b^{r+(s-s)} \\ &= b^s - b^{s+(r-s)} \\ &= b^s - b^s b^{r-s} \\ &= b^s(1 - b^{r-s}) \end{aligned}$$

Since $b > 1$ and $r - s > 0$, then $b^{r-s} > 1$ and $b^s > 0$. Thus,

$$b^s(1 - b^t) < 0$$

Therefore b^r is an upper bound of the set $B(r)$.
Now, consider $s > r$.

$$\begin{aligned} b^s - b^r &= b^{s+(r-r)} - b^r \\ &= b^r b^{s-r} - b^r \\ &= b^r(b^{s-r} - 1) \end{aligned}$$

Since $b > 1$ and $s - r > 0$, then $b^{s-r} > 1$ and $b^r > 0$. Thus,

$$b^r(b^{s-r} - 1) > 0$$

Therefore there is no $s > r$ in the set $B(r)$.
This means that we have proven

$$b^r = \sup B(r)$$

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

$$b^{x+y} = \sup B(x + y)$$

This means it is the supremum of the set of all numbers b^t, where t is rational and $r + s < x + y$.

Since every rational t, where $t < x + y$ can be written as the sum of two rationals r, s where $r + s = t$ and $r < x$ and $s < y$, then the set $B(x + y)$ is equivalent to the set $\{b^r b^s : (r, s) \in B(x) \times B(y)\}$.

Therefore, since $b > 1$ and all $b^x > 0$, then $\sup B(x + y) = \sup B(x) \sup B(y)$, and

$$b^{x+y} = b^x b^y$$

$\boxed{\text{Extra}}$ Induction Practice

Prove that $1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$ for all positive integers n.

(Base Case) It can easily be shown that $1^3 = 1^2 = 1$.

(Inductive Case) Assume that for some $n \in \mathbb{Z}^+, 1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$. Take $n + 1$, then

$$\begin{aligned} 1^3 + 2^2 + ... + n^3 + (n + 1)^3 &= (1 + 2 + ... + n)^2 + (n + 1)^3 \\ &= \left(\frac{n(n+1)}{2}\right)^2 + (n + 1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n + 1)^3 \\ &= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\ &= \frac{(n+1)^2(n^2 + 4(n+1)}{4} \\ &= \frac{(n+1)^2(n^2 + 4n + 4)}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4} \\ &= (1 + 2 + ... + n + (n + 1))^2 \end{aligned}$$

3

Therefore, by induction, $1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$ for all positive integers n.