9. f_n cont on E, $f_n \to f$ uniformly $\Longrightarrow$ f is cont.

Fix $x \in E$.

Given $\varepsilon > 0$, find δ s.t. $d(y,x) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2$.

Pick N_1 s.t. $n \geq N_1 \Rightarrow d(x_n, x) < \delta$.

Pick N_2 s.t. $n \geq N_2 \Rightarrow \| f_n - f \| < \varepsilon/2$

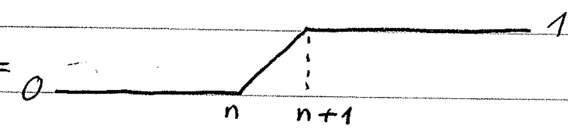
Let $N = N_1 \vee N_2$. Then $n \geq N \Rightarrow |f_n(x_n) - f(x)|$

$\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

This shows $f_n(x_n) \to f(x)$.

The converse is not true without further assumptions (such as compactness).

Example: $E = \mathbb{R}$, $f_n =$



Then $f_n \to f = 0$ but not uniformly.

Let $x_n \to x$ in $\mathbb{R}$. Then the set $\{x_n\}_{n \in \mathbb{N}} \cup \{x\}$ is bounded, say contained in $[-k, k]$.

For $n \geq k$, $f_n(x_n) = 0 = f(x)$.

18. $\|f_n\| \le M \quad \forall n.$

Claim: $\{F_n\}$ is uniformly bounded and equicontinuous.

Uniform bdd-ness: $|F_n(x)| \le \int_a^x |f_n(t)|\,dt \le M(x-a) \le M(b-a).$

Equicontinuity: Let $y<x$ in $[a,b]$. $|F_n(y)-F_n(x)| = \left|\int_x^y f_n\right|$
$\le M\cdot|y-x|.$

Thus given $\varepsilon>0$, let $\delta = \frac{\varepsilon}{M}$. Then $|x-y|<\delta$
$\Rightarrow |F_n(x)-F_n(y)| \le M|y-x| < M\delta = \varepsilon.$

Thm 7.25 $\Rightarrow \exists$ uniformly convergent subseq $\{F_{n_k}\}$.

20. $f \in C[0,1]$. Given $\varepsilon>0$, find polynomial p on $[0,1]$ s.t. $\|f-p\|<\varepsilon$. $p(x) = \sum_{j=0}^n a_j x^j$.

$$\int f^2 = \int f(f-p+p) = \int f(f-p) + \int fp$$

$$\le \|f\| \underbrace{\int |f-p|}_{<\varepsilon} + \sum_{j=0}^n a_j \underbrace{\int_0^1 f(x)x^j\,dx}_{=0} \le \varepsilon\|f\|.$$

Since $\varepsilon>0$ was arbitrary, $\int_0^1 f^2\,dx = 0$.
Appeal to Exercise 6.2 (from last hw) to conclude that $f^2(x)=0$ for all x, which implies $f=0$.

$$\int |fg| \, d\alpha \le \left(\int f^2 d\alpha \right)^{\frac{1}{2}} \left(\int g^2 \, d\alpha \right)^{\frac{1}{2}}$$

Using Schwarz ineq from Ch. 1:

$$\sum_i |f(s_i) g(s_i)| \Delta\alpha_i = \sum_i \left(|f(s_i)| \sqrt{\Delta\alpha_i} \right) \left(|g(s_i)| \sqrt{\Delta\alpha_i} \right)$$

$$\le \left(\sum_i |f(s_i)|^2 \Delta\alpha_i \right)^{1/2} \left(\sum_i |g(s_i)|^2 \Delta\alpha_i \right)^{1/2}$$

$$\le U(P, f^2, \alpha)^{1/2} \, U(P, g^2, \alpha)^{\frac{1}{2}} \le \left(\int f^2 d\alpha + \varepsilon \right)^{\frac{1}{2}} \left(\int g^2 d\alpha + \varepsilon \right)^{\frac{1}{2}}.$$

For the above:

Pick partitions P_1 and P_2 so that
$U(P_1, f^2, \alpha) \le \int f^2 d\alpha + \varepsilon$, $U(P_2, g^2, \alpha) \le \int g^2 d\alpha + \varepsilon$.

Let $P = P_1 \cup P_2$ be the common refinement.
Then also $U(P, f^2, \alpha) \le \int f^2 d\alpha + \varepsilon$, $U(P, g^2, \alpha) \le \int g^2 d\alpha + \varepsilon$.

Now choose a sequence of s_i's so that $|f(s_i) g(s_i)| \to \sup_{s \in [x_{i-1}, x_i]} |f(s) g(s)|$.

Then we have the inequality

$$\int |fg| \, d\alpha \le U(P, |fg|, \alpha) \le \left(\int f^2 d\alpha + \varepsilon \right)^{\frac{1}{2}} \left(\int g^2 d\alpha + \varepsilon \right)^{\frac{1}{2}}$$

Let $\varepsilon \downarrow 0$.