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Homework 10:

Rudin 7.9

$\{f_n\}$ is a sequence of continuous functions which converges uniformly to f on E.

Since unif. conv. implies pointwise convergence, $\forall \varepsilon > 0\ \exists N_1 \in \mathbb{N}$ such that $m \geq N_1 \Rightarrow d(f_m(x_n), f(x_n)) < \varepsilon/2$

By Thm 7.12, f has to be continuous on E. So $\forall \varepsilon > 0\ \exists N_2 \in \mathbb{N}$ such that $n \geq N_2 \Rightarrow d(f(x_n), f(x)) < \varepsilon/2 \quad \forall \{x_n\} \to x$

Thus, if $N = \max(N_1, N_2)$ then $\forall \varepsilon > 0$ and $n \geq N$, we have

$$d(f_n(x_n), f(x)) \leq d(f_n(x_n), f(x_n)) + d(f(x_n) - f(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Hence, $\forall \{x_n\} \to x$ in E, $\lim_{n \to \infty} f_n(x_n) = f(x)$

~~Rudin 7.18~~

Converse is not true always. Let $f_n(x) = \frac{1}{nx+1}$.

and define $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Then $\forall x_n \to x$, $f_n(x) \to f(x)$. But $\{f_n\}$ does not converge to f uniformly since ~~f is not continuous pointwise. it fails pointwise~~ f is not continuous.

Rudin 7.18

Since $\{f_n\}$ is uniformly bounded, let $|f_n(t)| < M < \infty$ for $t \in [a, b]$ and $n \in \mathbb{N}$.

$$\therefore |F_n(x)| = \left| \int_a^x f_n(t)\, dt \right| \leq \int_a^x |f_n(t)|\, dt < M(x-a) < \infty$$

$$\leq M(b-a)$$

for $x \in [a, b]$ and $n \in \mathbb{N}$.

Thus, $\{F_n(x)\}$ is pointwise bounded.

Now let $\varepsilon > 0$. Choose $\delta = \varepsilon / M$. So $\forall x, y \in [a,b]$ with $|x-y| < \delta$, we have

$$|F_n(x) - F_n(y)| = \left| \int_a^x f_n(t)\,dt - \int_a^y f_n(t)\,dt \right|$$
$$= \left| \int_y^x f_n(t)\,dt \right|$$
$$\leq \int_y^x |f_n(t)|\,dt$$
$$< M(x-y)$$
$$= M\delta$$
$$= \varepsilon$$

Thus, $\{F_n\}$ is equicontinuous on $[a,b]$

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Hence by Theorem 7.25(b), $\{F_n\}$ contains a uniformly convergent subsequence on $[a,b]$.

Rudin 7.20 f is continuous on $[0,1]$. Thus by Thm 7.26, $\exists \{P_n\}$ sequence of polynomials such that ~~$\{P_n(x)\} \to \{f(x)\}$~~ $\{P_n\} \to f$ uniformly. — ①

Also, f is bounded since it is continuous on a compact set. Let $|f(x)| \leq M < \infty$. — ②

From ①, for any $\varepsilon > 0$ we can choose $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |f(x) - P_n(x)| < \varepsilon / M$.

Now consider $n \geq N$ and $\varepsilon > 0$:

$$|f(x)P_n(x) - f^2(x)| = |f(x)|\,|P_n(x) - f(x)|$$
$$< M \cdot \frac{\varepsilon}{M}$$
$$= \varepsilon$$

Thus, $\{f.P_n\} \to f^2$ uniformly.

$\therefore$ By Thm 7.16,

$$\int_0^1 f^2(x)\,dx = \lim_{n\to\infty} \int_0^1 f(x) P_n(x)\,dx \quad -③$$

But $\int_0^1 f(x)P_n(x)dx = \int_0^1 f(x)\left(\sum_{i=0}^{n} a_i x^i\right)dx$

$$= \sum_{i=0}^{n} a_i \int_0^1 f(x)\, x^i\, dx$$

$$= 0 \quad \text{since} \quad \int_0^1 f(x)x^i dx = 0 \ \forall i$$

$\therefore$ ③ $\Rightarrow \int_0^1 f^2(x)\,dx = 0$

Using the result proved in last homework (exercise 6.2), we get $f^2(x) = 0 \quad \forall x \in [0,1]$

$\Rightarrow f(x) = 0 \quad \forall x \in [0,1]$.

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Extra Problem: We know that $\int_a^b [f - cg]^2\, d\alpha \geq 0 \quad \forall c \in \mathbb{R}$

since the integrand is always positive. $\Rightarrow \int_a^b f^2 d\alpha - 2c\int_a^b |fg| d\alpha + c^2 \int_a^b g^2 d\alpha \geq 0 \quad \to (*)$

(1) If $\int_a^b g^2 d\alpha \neq 0$, then we can put $c = \dfrac{\int_a^b |fg|\, d\alpha}{\int_a^b g^2 d\alpha}$ in $(*)$

$$\therefore \int_a^b f^2 d\alpha - 2\frac{\left(\int_a^b |fg|\, d\alpha\right)^2}{\int_a^b g^2 d\alpha} + \frac{\left(\int_a^b |fg|\, d\alpha\right)^2}{\int_a^b g^2 d\alpha} \geq 0$$

$$\Rightarrow \int_a^b f^2 d\alpha \cdot \int_a^b g^2 d\alpha \geq \left(\int_a^b |fg|\, d\alpha\right)^2$$

(2) If $\int_a^b g^2 \, d\alpha = 0$ then (*) becomes

$$\int_a^b f^2 \, d\alpha - 2c \int_a^b |fg| \, d\alpha \geq 0$$

$$\Rightarrow \int_a^b |fg| \, d\alpha \leq \frac{\int_a^b f^2 \, d\alpha}{2c}$$

$$\Rightarrow \lim_{c \to \infty} \int_a^b |fg| \, d\alpha \leq \lim_{c \to \infty} \frac{\int_a^b f^2 \, d\alpha}{2c}$$

$$\Rightarrow \int_a^b |fg| \, d\alpha \leq 0 = \int_a^b f^2 \, d\alpha \cdot \int_a^b g^2 \, d\alpha$$

$\square$