521 Analysis I Spring 2011 Final Exam

Keep justification short and to the point. The total is 100.

1. (10 pts) Let $(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ be the open unit interval in $\mathbb{R}$. Give an example of a sequence $\{x_n\}$ in $(0,1)$ such that $\overline{\lim}\, x_n$ lies outside $(0,1)$ but $\underline{\lim}\, x_n$ lies inside $(0,1)$.

$$x_n = \begin{cases} 1 - \frac{1}{n}, & n \text{ even} \\ \frac{1}{2}, & n \text{ odd} \end{cases}$$

2. (10 pts) Can there exist a continuous function f on $[0,1]$ such that f is not constant and all values $f(x)$ are rational?

No. $f([0,1])$ must be a connected set.

Suppose $u < v$ both lie in $f([0,1])$.

Then also $[u,v] \subseteq f([0,1])$, and there must be an irrational in (u,v).

3. (15 pts) Let (X, d) be a metric space and fix a point $w \in X$. Use the triangle inequality to show that the function $f(x) = d(x, w)$ is uniformly continuous.

$d(x,w) \leq d(x,y) + d(y,w) \Rightarrow d(x,w) - d(y,w) \leq d(x,y)$

$d(y,w) \leq d(y,x) + d(x,w) \Rightarrow d(y,w) - d(x,w) \leq d(x,y).$

Together these give $|d(x,w) - d(y,w)| \leq d(x,y).$

Thus, given $\varepsilon > 0$, if $d(x,y) < \varepsilon$ then

$|d(x,w) - d(y,w)| < \varepsilon.$

4. (15 pts) Let (X, d) be a metric space. Recall the definition of the distance of a point x to a set A:

$$\mathrm{dist}(x, A) = \inf\{d(x, a) : a \in A\}.$$

Suppose A is compact. Show that then there exists a point $z \in A$ such that $d(x, z) = \mathrm{dist}(x, A)$. ($z$ is a closest point to x in A.)

By #3, $f(a) = d(x,a)$ is a continuous function on A. A compact $\Rightarrow$ f attains its minimum at some $z \in A$. Then

$d(x,z) \leq d(x,a) \quad \forall a \in A.$

5. (a) (5 pts) Suppose f is a function defined on some infinite interval of the type $[a, \infty)$ and c is a real number. Give a precise definition for the limit $\lim_{x\to\infty} f(x) = c$.

(b) (15 pts) Let g be a continuous function on $[0, \infty)$ such that $\lim_{x\to\infty} g(x) = c$. Let

$$f(t) = \frac{1}{t}\int_0^t g(x)\,dx \qquad \text{for } t > 0.$$

Find the limit of $f(t)$ as $t \to \infty$ and prove this limit.

(a) $\forall \varepsilon > 0 \quad \exists M < \infty \quad s.t. \quad x > M \Rightarrow |f(x) - c| < \varepsilon.$

(b) $|f(t) - c| = \left| \frac{1}{t}\int_0^t g(x)\,dx - c \right| \leq \frac{1}{t}\int_0^t |g(x) - c|\,dx$

$= \frac{1}{t}\int_0^b |g(x) - c|\,dx + \frac{1}{t}\int_b^t |g(x) - c|\,dx.$

1. Pick $b < \infty$ s.t. $x \geq b \Rightarrow |g(x) - c| < \frac{\varepsilon}{2}$.

2. Pick $M \in (b, \infty)$ s.t. $t \geq M \Rightarrow \frac{1}{t}\int_0^b |g(x) - c|\,dx < \frac{\varepsilon}{2}$.

Then $t \geq M$ from above

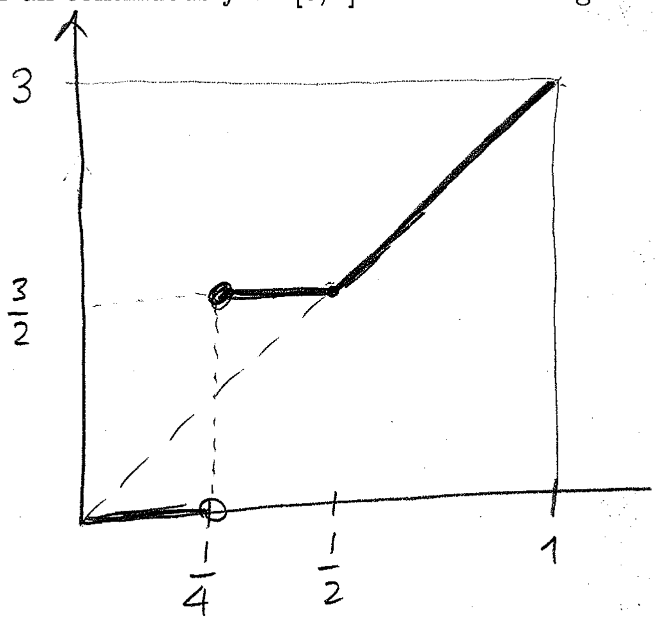
$$|f(t) - c| \leq \frac{\varepsilon}{2} + \frac{1}{t}\int_b^t \frac{\varepsilon}{2}\,dx < \varepsilon.$$

Conclusion: $\lim_{t\to\infty} f(t) = c.$

6. (10 pts) Let α be a nondecreasing function on $[0,1]$. Suppose that we know $\alpha(0) = 0$ and

$$\int_0^1 f\,d\alpha = \tfrac{3}{2} f(\tfrac{1}{4}) + 3\int_{1/2}^1 f(x)\,dx$$

for all continuous f on $[0,1]$. Based on this give a formula for α, or draw a precise graph for it.



α must have a jump of magnitude $3/2$ at $x = \frac{1}{4}$, and slope 3 on $[\frac{1}{2}, 1]$.

$$\alpha(x) = \begin{cases} 0, & x < \frac{1}{4} \\ 3/2, & \frac{1}{4} \le x < \frac{1}{2} \\ 3x, & \frac{1}{2} \le x \end{cases}$$

7. (a) (5 pts) State the definition of equicontinuity of a sequence of real-valued functions $\{f_n\}$ on a metric space.

(b) (15 pts) Let $\{f_n\}$ be a sequence of real-valued functions on $[0,1]$ such that each f_n is continuous on $[0,1]$, differentiable on $(0,1)$, $f_n(0)=0$, and $|f_n'(x)| \le 7$ for all n and all $x \in (0,1)$. Prove that there exists a subsequence of $\{f_n\}$ that converges uniformly on $[0,1]$.

(a) Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $d(x,y) < \delta$ implies $|f_n(x) - f_n(y)| < \varepsilon$ for all f_n.

(b) The theorem we need to use:
If $\{f_n\}$ is equicontinuous and pointwise bounded, then $\exists$ subseq $\{f_{n_k}\}$ that converges uniformly on $[0,1]$.

<u>Equicontinuity</u>: $\forall x < y$, by the MVT,
$|f_n(x) - f_n(y)| = |f_n'(\xi)(x-y)| \le 7|x-y|$.
Thus given $\varepsilon > 0$, $\delta = \frac{\varepsilon}{7}$ will work.

<u>Pointwise bounded</u>: From above,
$|f_n(x)| = |f_n(x) - f_n(0)| \le 7|x| \le 7$.