N = 28

# 521 Analysis I Spring 2011 Exam 3

Be sure to justify your answers. The point total is 100.

**1.** Let $f : [a, b] \to \mathbb{R}$ be continuous, and for $x \in [a, b]$ define another function

$$g(x) = \sup\{f(t) : t \in [a, x]\}.$$

(a) (20 pts) Does there exist for each $x \in [a, b]$ a point $t(x) \in [a, x]$ such that $g(x) = f(t(x))$? Explain.

(b) (20 pts) Prove that $g(x+) = g(x)$, in other words, that $g$ is right-continuous.

(a) Yes. $f$ continuous and $[a, x]$ compact $\Longrightarrow$ $\exists$ point $t \in [a, x]$ where $f$ achieves its maximum.

(b) The def shows $g$ monotone increasing, and so $g(x+)$ exists and $g(x+) \geq g(x)$.

Given $\varepsilon > 0$, choose $\delta > 0$ s.t. $|s - x| \leq \delta$ $\Longrightarrow$ $|f(s) - f(x)| \leq \varepsilon$.

Then $\sup_{t \in [x, x+\delta]} f(t) \leq f(x) + \varepsilon$, and consequently

for $y \in [x, x+\delta]$, $g(y) = \max\left\{ \sup_{t \in [a,x]} f(t), \sup_{t \in [x,y]} f(t) \right\}$

$\leq \max\{ g(x), f(x) + \varepsilon \} \leq g(x) + \varepsilon$.

This shows that $g(x+) \leq g(x) + \varepsilon$.

15 **2.** (~~10~~ pts) (a) State the definition of uniform continuity of a function $f : X \to Y$ where $X$ and $Y$ are metric spaces.

25 (b) (~~20~~ pts) Is the following function $f$ uniformly continuous on the interval $(0, \infty)$:

$$f(x) = x + \frac{1}{1+x}.$$

(a) $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. for all $x, x' \in X$, $d(x, x') < \delta$ implies $d(f(x), f(x')) < \varepsilon$.

(b) $$|f(x) - f(y)| = \left| x + \frac{1}{1+x} - y - \frac{1}{1+y} \right|$$

$$= \left| x - y + \frac{1}{1+x} - \frac{1}{1+y} \right| = \left| x - y + \frac{y - x}{(1+x)(1+y)} \right|$$

$$\leq |x-y| + \frac{|x-y|}{(1+x)(1+y)} \leq |x-y| + |x+y| = 2|x-y|.$$

So answer is YES; this function is even Lipschitz continuous.

(Given $\varepsilon > 0$, suppose $|x-y| < \frac{\varepsilon}{2}$. Then computation above shows $|f(x) - f(y)| < \varepsilon$.)

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**3.** (a) (~~10~~ pts) State the Mean Value Theorem precisely.

25 (b) (~~20~~ pts) Suppose $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, and $f'(x) \neq 0$ for all $x \in (a, b)$. Does it follow that $f$ is one-to-one?

(a) If $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then $\exists$ point $\xi \in (a,b)$ s.t. $f(b) - f(a) = f'(\xi)(b-a)$.

(b) Let $x < y$ in $[a,b]$. Then $\exists\, t \in (x,y)$ s.t. $f(y) - f(x) = f'(t)(y-x)$. Since $f'(t) \neq 0$ and $y - x > 0$, also $f(y) - f(x) \neq 0$.

So $f$ is one-to-one.