521 Analysis I Spring 2011 Exam 2

Be sure to justify your answers. The point total is 100.

1. (40 pts) Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space (X, d). Assume that $\{x_n\}$ is a Cauchy sequence and that $d(x_n, y_n) \to 0$ as $n \to \infty$. State the definition of a Cauchy sequence, and then prove that $\{y_n\}$ is also a Cauchy sequence.

$\{x_n\}$ is Cauchy if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that if $m, n \geq N$ then $d(x_m, x_n) \leq \varepsilon$.

To show $\{y_n\}$ Cauchy:

Pick N_1 s.t. $m, n \geq N_1 \Rightarrow d(x_m, x_n) \leq \frac{\varepsilon}{3}$.

Pick N_2 s.t. $n \geq N_2 \Rightarrow d(x_n, y_n) \leq \frac{\varepsilon}{3}$.

Now if $m, n \geq N_1 \vee N_2$ $(= \max\{N_1, N_2\})$

then $d(y_m, y_n) \leq d(y_m, x_m) + d(x_m, x_n) + d(x_n, y_n)$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

2. (20 pts) Let $\{a_n\}_{n\in\mathbb{Z}_+}$ be a sequence of real numbers. Assume that the series

$$\sum_{n=1}^{\infty} |a_n - a_{n-1}|$$

converges. Show that then the sequence $\{a_n\}$ converges to a limit in $\mathbb{R}$.

$$a_n = a_0 + a_n - a_0 = a_0 + \underbrace{\sum_{k=1}^{n} (a_k - a_{k-1})}$$

converges to a limit $\bar{a}$ in $\mathbb{R}$ because it is assumed to converge absolutely.

Thus $a_n = a_0 + \sum_{k=1}^{n} (a_k - a_{k-1}) \longrightarrow a_0 + \bar{a}$.

Alternative: Cauchy argument.

$\sum |a_n - a_{n-1}|$ converges, so given $\varepsilon > 0$ $\exists N$ s.t.

$q \geq p \geq N \Rightarrow \sum_{n=p}^{q} |a_n - a_{n-1}| < \varepsilon$.

Thus $q \geq p \geq N \Rightarrow |a_q - a_p| = \left| \sum_{n=p+1}^{q} (a_n - a_{n-1}) \right|$

$$\leq \sum_{n=p+1}^{q} |a_n - a_{n-1}| < \varepsilon.$$

3. (20 pts) Let $0 < x < 1$. Use any of our convergence/divergence criteria to decide and justify whether the series

$$\sum_{n=1}^{\infty} nx^n$$

converges or diverges.

MANY TESTS SHOW CONVERGENCE:

1. As a power series: radius of convergence $R = \dfrac{1}{\overline{\lim}\, n^{\frac{1}{n}}} = 1$, so converges for all $|x| < 1$.

2. Root test: $\overline{\lim}\, |nx^n|^{\frac{1}{n}} = x \cdot \overline{\lim}\, n^{\frac{1}{n}} = x < 1$
(We proved in class that $n^{\frac{1}{n}} \to 1$.)

3. Ratio test: $\overline{\lim} \left| \dfrac{(n+1)x^{n+1}}{nx^n} \right| = x \cdot \overline{\lim}\, \dfrac{n+1}{n} = x < 1$.

4. Comparison test: Let $0 < x < \beta < 1$.
Then $\dfrac{n}{(\beta/x)^n} \longrightarrow 0$, from which $nx^n \leq \beta^n$ for $n \geq N$, for some N. $\sum \beta^n$ converges as a geometric series, hence so does $\sum nx^n$.

4. (20 pts) Let $a_n \geq 0$. Suppose $\sum_{n=1}^{\infty} a_n$ converges. Does it follow that $\sum_{n=1}^{\infty} a_n^2$ converges?

YES. $\sum a_n$ converges $\Longrightarrow \exists N$ s.t. $0 \leq a_n \leq 1$ for $n \geq N$. Then $a_n^2 \leq a_n$ for $n \geq N$, and comparison test implies that $\sum a_n^2$ converges.

Cauchy argument: Given $\varepsilon > 0$, $\exists N$ s.t.

$$n \geqslant m \geqslant N \Longrightarrow \sum_{k=m}^{n} a_k < \sqrt{\varepsilon}.$$

(Since $a_k \geqslant 0$ no absolute values needed.)

Consequently

$$n \geqslant m \geqslant N \Longrightarrow \varepsilon \geqslant \left(\sum_{k=m}^{n} a_k\right)^2 = \sum_{k=m}^{n} a_k^2 + \sum_{\substack{m \leq k, \ell \leq n \\ k \neq \ell}} a_k a_\ell$$

$$\geqslant \sum_{k=m}^{n} a_k^2 .$$