521 Analysis I Spring 2011 Exam 1

Be sure to justify your answers. The point total is 100. Unless otherwise stated, the setting is a metric space (X, d).

1. (a) (10 pts) State the definition of a limit point and the closure $\bar{A}$ of a set A.

(b) (40 pts) The distance of a point x to a set A is defined as

$$\mathrm{dist}(x, A) = \inf\{d(x, a) : a \in A\}.$$

Show that if $\mathrm{dist}(x, A) = 0$ then $x \in \bar{A}$.

(b) $\mathrm{dist}(x, A) = 0 \implies \forall r > 0 \ \exists a \in A$ such that $a \in N_r(x)$.

Thus either x itself is a member of A, or every nbhd of x contains a point $a \in A$ s.t. $a \neq x$.

So either $x \in A$ or $x \in A'$, which says that $x \in \overline{A}$.

2. (a) (10 pts) State the definition of a compact set in a general metric space.

(b) (40 pts) Define the following subsets of the metric space $\mathbb{R}$:

$$B = \{\tfrac{1}{n} : n \in \mathbb{N}\} = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\} \quad \text{and} \quad A = B \cup \{0\}.$$

At least one of the sets A and B is compact. Pick one of them and show that it is compact by verifying the definition. (A solution by other means may get some partial credit.)

(b) To show A compact: Let $\{G_\alpha\}$ be an open cover of A.

Pick α_0 s.t. $0 \in G_{\alpha_0}$. Since G_{α_0} is open and contains 0, $\exists r > 0$ s.t. $(-r, r) \subseteq G_{\alpha_0}$.

Consequently G_{α_0} contains all $\frac{1}{n}$ for $n \geqslant n_0$ where n_0 is an integer chosen so that $n_0 > \frac{1}{r}$.

For $n = 1, 2, \ldots, n_0 - 1$ pick G_{α_n} so that $\frac{1}{n} \in G_{\alpha_n}$.

Now all points in A have been accounted for, so $A \subseteq G_{\alpha_0} \cup G_{\alpha_1} \cup \cdots \cup G_{\alpha_{n_0 - 1}}$.