AN EXPLICIT FORMULA FOR LOCAL DENSITIES OF QUADRATIC FORMS (J. NUMBER THEORY 72 (1998), NO. 2, 309–356) (SOME CORRECTION IN SECTION 7 IS INCLUDED)

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0. Introduction.

Let S and T be two positive definite integral matrices of rank m and n respectively. It is an ancient but still very challenging problem to determine how many times S can represent T, i.e., the number of integral matrices X with ${}^tXSX = T$. However, Siegel proved in his celebrated paper ([Si1]) that certain weighted averages of these numbers over the genus of S can be expressed as the Euler product of pure local data—confluent hypergeometric functions for $p = \infty$ and local densities $\alpha_p(T, S)$ for $p < \infty$ (see (1.1) for definition). Siegel himself extended this result to indefinite forms ([Si2-3]) in early fifties. A. Weil reinterpreted Siegel's results in terms of representation and extended his results to other classical groups in 1965 ([We]). Roughly speaking, the Siegel-Weil formula says that the theta integral associated to a vector space (quadratic or Hermitian) is the special value of some Eisenstein series at certain point when both the theta integral and Eisenstein series (at the point) are both absolutely convergent. Recently, Kudla and Rallis pushed the results to non-convergent regions ([KR1-3]). From the point view of representation theory, the local density can be viewed as the special value of a local Whittaker function, which is the local factor of the Fourier coefficients of the Eisenstein series. For a lot of arithmetic applications, it is very important to have an exact formula for the local densities. For example, in his work on central derivative of Eisenstein series ([Ku1]), Kudla needed to compare the local density of certain ternary form with intersection number on some formal group. The explicit formula for local density was also used in Gross and Keating's work in the intersection section of modular correspondence ([GK]). However, explicit formulas of local densities are known hard to obtain and are complicated in general. Siegel himself obtained an explicit formula for $n = 1$ or $m = n$ assuming S is unimodular and $p \neq 2$. Under the

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same assumption, Kitaoka obtained an explicit local density formula for $n = 2$ and some reduction formula in general in early eighties ([Ki1]). In 1991, Meyer generalized Kitaoka's result to cover the case where pS^{-1} is is integral. On the other hand, the case where S is hyperbolic, i.e., $S = \frac{1}{2}\begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$, has been extensively studied as related to the classical Siegel Eisenstein series and is more fruitful. Very recently, Katsurada gave a beautiful formula in this case with both n and p arbitrary ([Ka]). We refer to [Ki2] and [Ka] and its bibliography for further reference.

The purpose of this work is to give an explicit local density formula for arbitrary S and $n \le 2$, with one exception $n = 2 = p$. In other words, we will settle the 'ramified' case. Our method is quite different from others in this subject. Instead of using reduction and functional equation to reduce the problem to a simpler one and then computing the simpler one, we relate the local density to a (Whittaker) integral and then compute the integral directly. The results will be used to complete Kudla's work at ramified places. The local Whittaker integrals we computed are local factors of the Fourier Coefficients of Siegel Eisenstein series on $\mathrm{Sp}(n)$ and are of importance in their own right. It should be mentioned that the functional equation in the general case is very complicated and is not relating S to itself. The reduction formula is also absent in general and is only true after a certain stage (see Theorem 3.3). We should mention that the confluent hyper-geometric functions, the local problem at $p = \infty$, have been extensively studied by Shimura ([Sh]).

This paper is organized as follows. In section 1, we set up the notations and sketch the main formula. In section 2, we record formulas for a couple of simple Gauss type integrals which will be used in sections 3 and 7. In sections 3 ($p \ne 2$) and 4 ($p = 2$), we obtain a local density formula when T is a nonzero number, and derive some of its consequences, including reduction formula. In section 5, we prove an integral transformation formula which will translate the local Whittaker integral problem to a problem on Gauss integral over $\mathrm{GL}_2(\mathbb{Z}_p)$. In section 6, we give a complete solution to the Gauss integral over $\mathrm{GL}_2(\mathbb{Z}_p)$. Results in both sections 5 and 6 should have independent interests. It is also interesting to note that the Gauss integral just mentioned is related to rational points of certain elliptic curves over the finite field $\mathbb{F}_p$ in some case (Proposition 6.5). In section 7, we derive the main formula of the paper—an explicit local density formula for arbitrary S and $n = 2 \ne p$. In section 8, we apply the formulas obtained in sections 3, 4, and 7 to some interesting examples involving quaternions and obtain some interesting formulas. For example, we reproved a peculiar formula relating the derivative of the local density polynomial associated to the division quaternion algebra to the local density associated to the split quaternion algebra, first discovered by Kudla and Meyer (Me], see theorem 8.10).

This work was inspired by Kudla's paper ([Ku1]). The author thanks him for the inspiration and his continuing advice and encouragement. The author also thanks G. Prasad and D. Zagier for their valuable help during the preparation of this paper. Finally, the author would like to thank the Max-Planck Institut für Mathematik at Bonn for its hospitality and for providing the excellent working environment. The main part of this work was done during his stay at the MPI in the summer of 1997. The author thanks Kitaoka and the referee for their suggestions and comments.

1. General set-up and sketch of the main formula.

Let S and T be two integral nondegenerated symmetric matrices over $\mathbb{Z}_p$ of degree m and n respectively. We assume $m > n \geq 1$. Then the local density of representing T by S is defined by Siegel ([Si1]) as follows:

$$\alpha_p(T, S) = \lim_{t \longrightarrow \infty} (p^t)^{\frac{n(n+1)}{2} - mn} A_t(T, S), \tag{1.1}$$

where

$$A_t(T, S) = \#\{X \in M_{m,n}(\mathbb{Z}_p/p^t) : \, S[X] \equiv T \bmod p^t\}.$$

To a nondegenerate symmetric matrix S of degree m, we associate a quadratic space $V = \mathbb{Q}_p^m$ of columns of length m with quadratic form $q(x) = \frac{1}{2}(x, x) = {}^t xSx$. Let $L = \mathbb{Z}_p^m$ be its standard lattice. Notice that L and S uniquely determine each other up to $\mathbb{Z}_p$-equivalence. Let $\mathrm{Sym}_n(\mathbb{Q}_p)$ denote the group of symmetric matrices of degree n over $\mathbb{Q}_p$. Define

$$W(T, S) = \int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \int_{L^n} \psi(\, \mathrm{tr} b q(x)) \psi(- \, \mathrm{tr} T b) \, dx \, db. \tag{1.2}$$

Here the Haar measure dx and db are product measure of the 'standard' Haar measure on $\mathbb{Q}_p$ (i.e., $\mathrm{meas}(\mathbb{Z}_p) = 1$), and $\psi(x) = e^{-2\pi i \lambda(x)}$ is the 'canonical' character of $\mathbb{Q}_p$, where $\lambda : \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$. The starting point of this work is the following well-known formula ([Ku1, Appendix]):

$$\alpha_p(T, S) = W(T, S). \tag{1.3}$$

By (1.2) and (1.3), it is obvious that $\alpha_p(T, S)$ is only dependent on the $\mathbb{Z}_p$-equivalent classes of S and T. For an integer $r \geq 0$, set

$$S_r = S \perp \frac{1}{2} \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}. \tag{1.4}$$

Then $\alpha_p(T, S_r)$ is a polynomial of $X = p^{-r}$ and will be denoted in this paper by $\alpha(X, T, S)$. In particular, $\alpha(1, T, S) = \alpha_p(T, S)$. The local density polynomial $\alpha(X, T, S)$ is closely related to local Whittaker functionals (see for example [Ku1, appendix]). We assume throughout this paper that S is half-integral but $p^{-1}S$ is not, and set $S^l = p^l S$ for $0 \leq l \leq 1$. This assumption on S is not restrictive since $p^2 S$ and S correspond to the same quadratic space. The main formulas in this paper can be summarized roughly as follows:

Theorem. *Let notation and assumption be as above. Then there are explicitly constructed polynomials $R_i(X,T,S)$ such that the following are true.*

(1) *(Theorems 3.1, 4.1) When $n=1$, and p is any prime number, one has*

$$\alpha(X,T^l,S^l) = 1 + p^l X^l R_1(X,T,S) + (1-p^{-1}) l p^l X^l.$$

In particular,

$$\alpha(X,T^1,S^1) - pX\alpha(X,T,S) = 1 - X.$$

(2) *(Theorem 7.1) When $n=2$ and $p \neq 2$, one has*

$$\alpha(X,T^l,S^l) = 1 + p^{2l} X^l R_1(X,T,S) + p^{3l} X^{2l} R_2(X,T,S) + l\beta^l(X,T,S)$$

where

$$\beta^l(X,T,S) = (1-p^{-2}) p^{2l} X^l + (1-p^{-1}) p^{3l} X^{2l}(1 + R_1(X,T,S)).$$

In particular,

$$\alpha_p(X,T^1,S^1) - p^3 X^2 \alpha(X,T,S) = (1-X)(1 + p^2 X + p^2 X R_1(X)).$$

It is interesting to notice from theorem that the reduction formula from $\alpha(X,T^l,S^l)$ to $\alpha(X,T,S)$ is much easier than that of $\alpha(X,T,S)$ (R_2 is much more complicated than R_1 in general).

When $p \neq 2$, we may and will assume throughout this paper that S is $\mathbb{Z}_p$-equivalent to

$$\operatorname{diag}(\epsilon_1 p^{l_1}, \cdots, \epsilon_m p^{l_m}) \text{ with } \epsilon_i \in \mathbb{Z}_p^* \text{ and } l_1 \le l_2 \cdots \le l_m. \tag{1.5}$$

The above assumption on S means $l_1 = 0$. For each integer $k \ge 0$, set

$$L(k,1) = \{1 \le i \le m : \ l_i - k < 0 \text{ is odd } \}, \quad l(k,1) = \#L(k,1). \tag{1.6}$$

Furthermore, we define

$$d(k) = k + \frac{1}{2}\sum_{l_i < k}(l_i - k), \tag{1.7}$$

$$v(k) = (\frac{-1}{p})^{[\frac{l(k,1)}{2}]} \prod_{i \in L(k,1)} (\frac{\epsilon_i}{p}), \tag{1.8}$$

Finally, We define

$$\delta_p = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ \sqrt{-1} & \text{if } p \equiv -1 \bmod 4, \end{cases} \tag{1.9}$$

and

$$\gamma(t) = \begin{cases} 1 & \text{if } a \text{ is even}, \\ \delta_p(\frac{-\alpha}{p}) & \text{if } a \text{ is odd}. \end{cases} \tag{1.10}$$

for $t = \alpha p^a$ with $\alpha \in \mathbb{Z}_p^*$ and $a \in \mathbb{Z}$. Here (and throughout this paper) we write $(\frac{\alpha}{p})$ for the Hilbert symbol $(\alpha, p)_p$.

2. Gauss integrals over $\mathbf{GL}_1$.

In this section, we assume $p \ne 2$ and compute the following four integrals needed later.

$$I(t) = \int_{\mathbb{Z}_p} \psi(tx^2)dx, \quad t \in \mathbb{Q}_p^*, \tag{2.1}$$

$$I^*(t) = \int_{\mathbb{Z}_p^*} \psi(tx^2)dx, \quad t \in \mathbb{Q}_p^*, \tag{2.2}$$

$$J(\beta p^b) = \int_{\mathbb{Z}_p^m} \psi(\beta p^b q(x))dx, \quad \beta \in \mathbb{Z}_p^*, b \in \mathbb{Z}, \tag{2.3}$$

$$I(t, \chi) = \int_{\mathbb{Z}_p^*} \chi(x)\psi(tx)dx, \quad t \in \mathbb{Q}_p^*. \tag{2.4}$$

Here $q(x)$ the quadratic form associated to S as in section 1, and χ is a character of $\mathbb{Q}_p^*$. Here the Haar measure on $\mathbb{Q}_p^*$ is the restriction of the Haar measure on $\mathbb{Q}_p$.

Lemma 2.1. *Let $t = \alpha p^a \in \mathbb{Q}_p^*$ with $\alpha \in \mathbb{Z}_p^*$ and $a \in \mathbb{Z}$.*

(1) *Let $\gamma(t)$ be as in (1.10). Then*

$$I(t) = \begin{cases} 1 & \text{if } a \ge 0, \\ p^{\frac{a}{2}}\gamma(t) & \text{if } a < 0. \end{cases}$$

(2)

$$I^*(t) = I(t) - p^{-1}I(p^2t) = \begin{cases} 1 - p^{-1} & \text{if } a \ge 0, \\ p^{-\frac{1}{2}}\delta_p(\frac{-\alpha}{p}) - p^{-1} & \text{if } a = -1, \\ 0 & \text{if } a \le -2. \end{cases}$$

In particular, $I(t) = p^{-1}I(p^2t)$ when $ord_p t \le -2$.

(3) *For any $\beta \in Z_p^*$, one has*

$$\int_{\beta+p\mathbb{Z}_p} \psi(tx^2)dx = p^{-1}\psi(t\beta^2)\ char(p^{-1}\mathbb{Z}_p)(t).$$

Here and throughout this paper, $char(X)$ stands for the characteristic function of a set X.

Proof. (1). It is obvious for $a \ge 0$. Assume $a < 0$. Let $k = [\frac{-a+1}{2}]$ be the integral part of $\frac{-a+1}{2}$. Then

$$\begin{aligned} I(t) &= \sum_{y \in \mathbb{Z}_p/p^k} \int_{y+p^k\mathbb{Z}_p} \psi(tx^2)dx \\ &= \sum_{y \in \mathbb{Z}_p/p^k} p^{-k}\psi(ty^2) \int_{\mathbb{Z}_p} \psi(2\alpha p^{a+k}yx)dx \\ &= p^{-k} \sum_{y \in \mathbb{Z}_p/p^k} \psi(ty^2)\ \mathrm{char}(p^{-a-k}\mathbb{Z}_p)(y) \end{aligned}$$

When a is even, $a = -2k$ and $-a-k = k$. So the only nonzero term in last integral is the one with $y = 0$. Therefore $I(t) = p^{-k} = p^{a/2}$. When a is odd, $a = -2k+1$ and $-a-k = k-1$. So

$$\begin{aligned} I(t) &= p^{-k} \sum_{y \in \mathbb{Z}_p/p} \psi(tp^{2k-2}y^2) \\ &= p^{-k} \sum_{y \in \mathbb{Z}/p} e^{\frac{-2\pi i \alpha y^2}{p}} \\ &= p^{-k}\sqrt{p}\delta_p(\frac{-\alpha}{p}) \end{aligned}$$

as expected, the final step is a well-known Gauss sum formula. Claim (2) follows from (1) easily. To prove (3), one first notices that

$$\int_{\beta+p\mathbb{Z}_p} \psi(tx^2)dx = p^{-1}\psi(t\beta^2) \int_{\mathbb{Z}_p} \psi(2\beta tpx)\psi(tp^2x^2)dx.$$

Write $t = \alpha p^a$ with $\alpha \in \mathbb{Z}_p^*$ and $a \in \mathbb{Z}$. When $a \ge -1$,

$$\int_{\mathbb{Z}_p} \psi(2\beta tpx)\psi(tp^2x^2)dx = \int_{\mathbb{Z}_p} dx = 1$$

as expected. When $a < -1$, we have to prove that the integral vanishes. Substitute x by $x + p^{-a-2}$, one has

$$\int_{\mathbb{Z}_p} \psi(2\beta tpx)\psi(tp^2x^2)dx = \psi(2\alpha\beta p^{-1}) \int_{\mathbb{Z}_p} \psi(2\beta tpx)\psi(tp^2x^2)dx.$$

Since $\psi(2\alpha\beta p^{-1}) \ne 1$, one sees that the integral has to be zero.

As for integral (2.3), we may assume that $q(x) = \sum \epsilon_i p^{l_i} x_i^2$ by (1.5). So Lemma 2.1 implies

Lemma 2.2. *With notation as above, one has*

$$J(\beta p^{-k}) = ((\frac{-\beta}{p})\delta_p)^{l(k,1)} \prod_{i \in L(k,1)} (\frac{\epsilon_i}{p}) p^{\frac{1}{2}\sum_{l_i<k} l_i - k}.$$

In particular, $J(\beta p^{-k}) = 1$ *when* $k \le 0$.

Corollary 2.3. *Notation as above.*

(1) $J(\beta p^{-k}) = (\frac{\beta}{p})^{l(k,1)} J(p^{-k})$.

(2) *If* $l_m < k$ *then*

$$J(p^{-k}) = \delta_p^{3l(k,1)} \prod_{i \in L(k,1)} (\frac{\epsilon_i}{p}) p^{\frac{1}{2}\sum_{l_i<k} l_i - k}.$$

Here we have used the fact that $\delta_p^2 = (\frac{-1}{p})$. The following is just an integral version of a well-known fact on Gauss sums.

Lemma 2.4. *Let* $t = \alpha p^a$ *with* $\alpha \in \mathbb{Z}_p^*$ *and* $a \in \mathbb{Z}$, *and let* χ *be a character of* $\mathbb{Q}_p^*$ *of conductor* $n = n(\chi)$. *Then*

$$I(t,\chi) = \begin{cases} 1 - p^{-1} & \text{if } a \ge 0 = n, \\ p^a \chi(-\alpha) G(\chi) & \text{if } n = -a, \\ 0 & \text{otherwise.} \end{cases}$$

Here

$$G(\chi) = \sum_{x \in \mathbb{Z}/p^n} \chi(x)\psi(x)$$

is the Gauss sum of χ *with respect to* ψ. *In particular, one has*

$$I(t, (\frac{\ }{p})) = p^{-\frac{1}{2}} \delta_p (\frac{-\alpha}{p}) \ \mathit{char}(p^{-1}\mathbb{Z}_p^*)(t).$$

3. The case $n = 1$ and $p \neq 2$.

In this section, we assume that $n = 1$ and $p \neq 2$. Let the notation be as in section 1. For $t = \alpha p^a$ with $\alpha \in Z^*$ and $a \in \mathbb{Z}$, set

$$f_1(t) = \begin{cases} -\frac{1}{p} & \text{if } l(a+1,1) \text{ is even,} \\ (\frac{\alpha}{p})\frac{1}{\sqrt{p}} & \text{if } l(a+1,1) \text{ is odd.} \end{cases} \tag{3.1}$$

and

$$R_1(X,t,S) = (1-p^{-1}) \sum_{\substack{0<k\le a \\ l(k,1) \text{ is even}}} v_k p^{d(k)} X^k + v_{a+1} p^{d(a+1)} f_1(t) X^{a+1}. \tag{3.2}$$

Theorem 3.1. *Let the notation be as above and assume $a \geq 0$. Then $(0 \leq l \leq 1)$*

$$\alpha(X, p^l t, S^l) = 1 + p^l X^l R_1(X, t, S) + (1 - p^{-1}) l p^l X^l.$$

Proof. First we assume $l = r = 0$. So $\alpha(1, t, S) = \alpha_p(t, S)$ (recall $X = p^{-r}$). By (1.3) and lemma 2.2, one has

$$\begin{aligned}
\alpha_p(t, S) &= W(t, S) \\
&= \int_{\mathbb{Q}_p} J(b)\psi(-tb)db \\
&= 1 + \sum_{k>0} p^k \int_{\mathbb{Z}_p^*} J(\beta p^{-k})\psi(-\alpha\beta p^{a-k})d\beta \\
&= 1 + \sum_{\substack{k>0 \\ l(k,1) \text{ even}}} p^{d(k)} \delta_p^{3l(k,1)} \prod_{i \in L(k,1)} (\frac{\epsilon_i}{p}) \int_{\mathbb{Z}_p^*} \psi(-\alpha\beta p^{a-k})d\beta \\
&\quad + \sum_{\substack{k>0 \\ l(k,1) \text{ odd}}} p^{d(k)} \delta_p^{3l(k,1)} \prod_{i \in L(k,1)} (\frac{\epsilon_i}{p}) \int_{\mathbb{Z}_p^*} (\frac{\beta}{p})\psi(-\alpha\beta p^{a-k})d\beta
\end{aligned}$$

Here δ_p is the number defined by (1.9). Notice that $\delta_p^2 = (\frac{-1}{p})$ and thus

$$v_k = \begin{cases} \delta_p^{3l(k,1)} \prod_{i \in L(k,1)} (\frac{\epsilon_i}{p}) & \text{if } l(k,1) \text{ is even,} \\ \delta_p^{3l(k,1)+1} \prod_{i \in L(k,1)} (\frac{\epsilon_i}{p}) & \text{if } l(k,1) \text{ is odd.} \end{cases}$$

So one has by Lemma 2.4

$$\begin{aligned}
\alpha_p(t, S) &= 1 + \sum_{\substack{k>0 \\ l(k,1) \text{ even}}} v_k p^{d(k)} (\operatorname{char}(p^k \mathbb{Z}_p) - \frac{1}{p} \operatorname{char}(p^{k-1}\mathbb{Z}_p))(t) \\
&\quad + \sum_{\substack{k>0 \\ l(k,1) \text{ odd}}} v_k p^{d(k)} (\frac{\alpha}{p}) p^{-\frac{1}{2}} \operatorname{char}(p^{k-1}\mathbb{Z}_p^*)(t) \\
&= 1 + (1 - \frac{1}{p}) \sum_{\substack{0<k\leq a \\ l(k,1) \text{ even}}} v_k p^{d(k)} + v_{a+1} f_1(t) p^{d(a+1)} \\
&= 1 + R_1(1, t, S).
\end{aligned}$$

This proves the case $l = r = 0$. Since $\alpha_p(pt, pS) = p\alpha_p(t, S)$, the formula is also true for $l = 1$ and $r = 0$. In general, denote $L_r^l(k, 1)$ and so on for the data corresponding to S_r^l instead of S. Then for any $k \geq 0$, one has

$$\begin{aligned}
l_r^l(k + l, 1) &\equiv l(k, 1) \bmod 2, \\
v_r^l(k + l) &= v(k),
\end{aligned}$$

and

$$d_r^l(k+l) = d(k) - rk + l - rl.$$

So the associated function f_1 is independent of r or l. Now applying the formula just proved to $p^l t$ and S_r^l, one gets the desired formula in the general case.

When $l_i = 0$, i.e., S is unimodular, and $l = r = 0$, Theorem 3.1 recovers the well-known formula of Siegel on local densities ([Si1]). We remark that the formula is also true for $t = 0$, i.e., $a = \infty$.

Corollary 3.2**.** *Let the notation and assumption be as in Theorem* 3.1*. Then*

$$\alpha(X, pt, pS) - pX\alpha(X, t, S) = 1 - X.$$

Theorem 3.3 (Induction formula). *Let the notation be as in Theorem* 3.1*. Then for* $ord_p t \geq l_m + 2 + l$*, one has*

$$\alpha(X, p^2 t, S^l) - \alpha_p(X, t, S^l) = p^{2-m}(\alpha_p(X, t, S^l) - \alpha_p(X, p^{-2}t, S^l)).$$

Furthermore, if m *is even, one has for* $ord_p t \geq l_m + 1 + l$

$$\alpha(X, pt, S^l) - \alpha(X, t, S^l) = \nu p^{1-\frac{m}{2}}(\alpha(X, t, S^l) - \alpha(X, p^{-1}t, S^l)).$$

Here $\epsilon_S = \prod \epsilon_i$ *and* $\nu = ((-1)^{\frac{m}{2}}\epsilon_S, p)_p$*.*

Proof. It is enough to prove the special case $l = r = 0$. The general case follows when one applies the special case to $(p^l t, S_r^l)$. Let $t = \alpha p^a$ with $a \geq l_m$. First notice that $d(k+1) = d(k) + 1 - \frac{m}{2}$ for $k > l_m$. One has by Theorem 3.1

$$\begin{aligned}
&\alpha_p(tp^2, S) - \alpha_p(t, S) \\
&= (1 - \frac{1}{p}) \sum_{\substack{a+1 \leq k \leq a+2 \\ l(k,1) \text{ even}}} v_k p^{d(k)} + v_{a+3} f_1(tp^2) p^{d(a+3)} - v_{a+1} f_1(t) p^{d(a+1)} \\
&= p^{d(a+1)}\{(1 - \frac{1}{p}) \sum_{\substack{0 \leq k \leq 1 \\ l(k+a+1,1) \text{ even}}} v_{a+1+k} p^{k(1-\frac{m}{2})} \\
&\qquad\qquad + v_{a+3} f_1(\alpha p^{a+2}) p^{2(1-\frac{m}{2})} - v_{a+1} f_1(\alpha p^a)\}.
\end{aligned}$$

Notice that v_k, $l(k, 1)$, and $f_1(\alpha p^{k-1})$ depend only on the parity of k when $k > l_m$. So the expression in the brace does not change when we replace a by $a + 2$. Therefore

$$\alpha_p(tp^4, S) - \alpha_p(tp^2, S) = p^{2-m}(\alpha_p(tp^2, S) - \alpha_p(t, S))$$

when $a =$ $\mathrm{ord}_p t \ge l_m$. This proves the first induction formula.

Now assume that m is even. Again write $t = \alpha p^a$ with $a \ge l_m$. There are two cases. If $l(a+1,1)$ is even, so is $l(a+2,1) = m - l(a+1,1)$ (thanks to $a+1 > l_m$). So

$$\begin{aligned}
&\alpha_p(tp, S) - \alpha_p(t, S) \\
&= (1 - \frac{1}{p}) v_{a+1} p^{d(a+1)} - \frac{1}{p} v_{a+2} p^{d(a+2)} + \frac{1}{p} v_{a+1} p^{d(a+1)} \\
&= v_{a+1} p^{d(a+1)} (1 - \nu p^{-\frac{m}{2}}).
\end{aligned}$$

Here we have used the fact that when m is even

$$\frac{v_{a+1}}{v_{a+2}} = v_{a+1} v_{a+2} = \nu. \tag{3.3}$$

Because of (3.3), it is now obvious that

$$\alpha_p(tp^2, S) - \alpha_p(tp, S) = \nu p^{1-\frac{m}{2}} (\alpha_p(tp, S) - \alpha_p(t, S))$$

for $a \ge l_m$. The case where $l(a+1,1)$ is odd is similar and left to the reader.

4. The case $n = 1$ and $p = 2$.

In this section, we consider the case $n = 1$ and $p = 2$. By [Ca, Lemma 8.4.1], a nonsingular symmetric matrix S over $\mathbb{Q}_2$ is $\mathbb{Z}_2$ equivalent to

$$\mathrm{diag}(\epsilon_1 2^{l_1}, \cdots, \epsilon_L 2^{l_L}) \oplus \left(\oplus_{i=1}^{M} \epsilon_i' 2^{m_i} \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \right) \oplus \left(\oplus_{j=1}^{N} \epsilon_j'' 2^{n_j} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \right) \tag{4.1}$$

where $\epsilon_h, \epsilon_i', \epsilon_j'' \in \mathbb{Z}_2^*$, l_h, m_i, and n_j are all integers. The restriction on S (section 1) means that the smallest integer among all l_h, m_i, and n_j is zero. Write $S^l = 2^l S$ as before for $l = 0$ or 1. Notice that $L + 2M + 2N = m$ is the degree of S. The corresponding lattice has a basis over which the corresponding quadratic form is given by

$$q = \sum_{h=1}^{L} \epsilon_h 2^{l_h} x_h^2 + \sum_{i=1}^{M} \epsilon_i' 2^{m_i} y_{i1} y_{i2} + \sum_{j=1}^{N} \epsilon_j'' 2^{n_j} (z_{j1}^2 + z_{j1} z_{j2} + z_{j2}^2). \tag{4.2}$$

For each integer $k>0$ we denote

(4.3)
$$\begin{aligned} L(k,1) &= \{l_h : l_h - k < 0 \text{ is odd }\}, \qquad l(k,1) = \#L(k,1), \\ p(k) &= (-1)^{\sum_{n_j<k}(n_j-k)} \\ \epsilon(k) &= \prod_{h\in L(k-1,1)} \epsilon_h \\ d(k) &= k + \frac{1}{2}\sum_{l_h<k-1}(l_h-k+1) + \sum_{m_i<k}(m_i-k) + \sum_{n_j<k}(n_j-k), \\ \delta(k) &= \begin{cases} 0 & \text{if } l_h = k-1 \text{ for some } h, \\ 1 & \text{otherwise .} \end{cases} \end{aligned}$$

Furthermore, we define for $t = \alpha 2^a$ with $\alpha \in \mathbb{Z}_2^*$ and $a \in \mathbb{Z}$

(4.4)
$$\begin{aligned} R_1(X,t,S) = & \sum_{\substack{l<k\le a+3 \\ l(k-1,1) \text{ odd}}} \delta(k)p(k)\left(\frac{2}{\mu\epsilon(k)}\right)2^{d(k)-\frac{3}{2}}X^k \\ &+ \sum_{\substack{l<k\le a+3 \\ l(k-1,1) \text{ even}}} \delta(k)p(k)\left(\frac{2}{\epsilon(k)}\right)2^{d(k)-1}\psi(\frac{\mu}{8})\ \mathrm{char}(4\mathbb{Z}_2))(\mu)X^k, \end{aligned}$$

where

$$(\frac{2}{x}) = \begin{cases} (2,x)_2 & \text{if } x \in \mathbb{Z}_2^*, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

and $\mu = \mu_k(t)$ is given by

$$\mu_k(t) = \alpha 2^{a-k+3} - \sum_{l_h<k-1} \epsilon_h \tag{4.6}$$

Theorem 4.1. *With notation as above, one has for $a \ge 0$*

$$\alpha(X, 2^l t, S^l) = 1 + 2^l X^l R_1(X,t,S) + lX^l.$$

In particular, one has

$$\alpha(X, 2t, 2S) - 2X\alpha(X,t,S) = 1 - X.$$

Again, theorem 4.1 is also true for $t=0$. We remark that the terms in the formula with $k \le a$ are independent of a. From this it is not difficult to derive a inductive formula for $p=2$ similar to Theorem 3.3. However, even when m is even, one does not have a reduction formula for $\alpha_2(2t,S) - \alpha_2(t,S)$ similar to that of Theorem 3.3 since there is not a good reduction formula for $\mu_k(t)$. In other words, we only have

Corollary 4.2. *Let the notation be as in Theorem 4.1. Then for $a \geq 4 + max(l_h, m_i, n_j)$, one has*

$$\alpha_2(4t, S) - \alpha_2(t, S) = 2^{2-m}\left(\alpha_2(t, S) - \alpha_2(\frac{1}{4}t, S)\right).$$

The proof of Theorem 4.1 is similar to that of Theorem 3.1. First we need two lemmas. The first one is similar to Lemmas 2.1 and 2.4 and the proof is left to the reader.

Lemma 4.3. (1) *For $\alpha \in \mathbb{Z}_2^*$ and $a \in \mathbb{Z}$, one has*

$$\begin{aligned}&\int_{\mathbb{Z}_2} \psi(\alpha 2^a x^2)dx \\ &= \begin{cases} 1 & \text{if } a \geq 0 \\ 0 & \text{if } a = -1 \\ 2^{\frac{a+1}{2}}\psi(\frac{\alpha}{8})(\frac{2}{\alpha})^{a+1} & \text{if } a < -1. \end{cases}\end{aligned}$$

(2) *Let χ be a character of $\mathbb{Q}_2^*$ with conductor $n(\chi)$. Then for α and a as above, one has*

$$\begin{aligned}&\int_{\mathbb{Z}_2^*} \chi(x)\psi(\alpha 2^a x)dx \\ &= \begin{cases} \frac{1}{2} & \text{if } n(\chi) = 0 \,\&\, a \geq 0 \\ -\frac{1}{2} & \text{if } n(\chi) = 0 \,\&\, a = -1 \\ 2^a\chi(-\alpha)G(\chi) & \text{if } n(\chi) = -a \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

In particular, one has

$$\int_{\mathbb{Z}_2^*} (\frac{2}{x})\psi(\alpha 2^a x)dx = \frac{1}{2\sqrt{2}}(\frac{2}{-\alpha})\delta_{-a,3}.$$

Here $\delta_{m,n}$ is the usual Kronecker symbol.

Lemma 4.4. *For $t = \alpha 2^a$ with $\alpha \in \mathbb{Z}_2^*$ and $a \in \mathbb{Z}$, one has*

$$\int_{\mathbb{Z}_2^2} \psi(ty_1y_2)dy_1\, dy_2 = min(1, 2^a) \tag{4.6}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}_2^2} \psi(t(z_1^2 + z_1z_2 + z_2^2)dz_1dz_2 \\ &= \begin{cases} 1 & \text{if } a \geq 0 \\ 2^a(-1)^a & \text{if } a < 0. \end{cases}\end{aligned} \tag{4.7}$$

Proof. Only (4.7) needs some verification for $a<0$. Set $k=[\frac{-a+1}{2}]$. Then

$$\begin{aligned}
&\int_{\mathbb{Z}_2}\psi(t(z_1z_2+z_2^2))dz_2\\
&=\sum_{x\in\mathbb{Z}/2^k}2^{-k}\psi(t(z_1x+x^2))\int_{\mathbb{Z}_2}\psi(\alpha 2^{a+k}(z_1+2x)z_2)dz_2\\
&=2^{-k}\sum_{x\in\mathbb{Z}/2^k}\psi(t(z_1x+x^2))\ \mathrm{char}(-2x+2^{-a-k}\mathbb{Z}_2)(z_1).
\end{aligned}$$

So

$$\begin{aligned}
&\int_{\mathbb{Z}_2^2}\psi(t(z_1^2+z_1z_2+z_2^2))dz_1dz_2\\
&=2^{-k}\sum_{x\in\mathbb{Z}/2^k}\int_{-2x+2^{-a-k}\mathbb{Z}_2}\psi(t(z_1^2+z_1x+x^2))dz_1\\
&=2^{a}\sum_{x\in\mathbb{Z}/2^k}\psi(3tx^2)\int_{\mathbb{Z}_2}\psi(-3x\alpha 2^{-k}z_1)\psi(\alpha 2^{-2k-a}z_1^2)dz_1.
\end{aligned}$$

When $a=-2k$ is even, the last integral is

$$\int_{\mathbb{Z}_2}\psi(-3x\alpha 2^{-k}z_1)dz_1=\ \mathrm{char}(2^k\mathbb{Z}_2)(x).$$

This proves the even case.

When $a=-2k+1$ is odd, the last integral is

$$\begin{aligned}
&\int_{\mathbb{Z}_2}\psi(-3x\alpha 2^{-k}z_1)\psi(\frac{\alpha}{2}z_1^2)dz_1\\
&=\int_{2\mathbb{Z}_2}\psi(-3x\alpha 2^{-k}z_1)dz_1-\int_{1+2\mathbb{Z}_2}\psi(-x\alpha 2^{-k}z_1)dz_1\\
&=\frac{1}{2}(1-\psi(-3x\alpha 2^{-k}))\ \mathrm{char}(2^{k-1}\mathbb{Z}_2)(x)\\
&=\ \mathrm{char}(2^{k-1}\mathbb{Z}_2^*)(x).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{Z}_2^2}\psi(t(z_1^2+z_1z_2+z_2^2))dz_1dz_2\\
&=2^a\psi(3\alpha 2^a 2^{2k-2})\\
&=2^a\psi(\frac{3\alpha}{2})=-2^a.
\end{aligned}$$

Proof of Theorem 4.1 Now the proof of Theorem 4.1 becomes a formality. Just as in the proof of Theorem 3.1, we may assume that $l = r = 0$. Assume $a \geq 0$. By Lemma 1.3, one has for $t = \alpha 2^a$

$$\begin{aligned}
&\alpha_2(t, S)) \\
&= \int_{\mathbb{Q}_2} \int_{\mathbb{Z}_2^m} \psi(bq(x)) dx \psi(-tb) db \\
&= 1 + \sum_{k=1}^{\infty} 2^k \int_{\mathbb{Z}_2^*} J_1(\beta 2^{-k}) J_2(\beta 2^{-k}) J_3(\beta 2^{-k}) \psi(-\alpha\beta 2^{a-k}) d\beta
\end{aligned}$$

where

$$\begin{aligned}
J_1(b) &= \prod_{h=1}^{L} \int_{\mathbb{Z}_2} \psi(\epsilon_h 2^{l_h} b x_h^2) dx \\
J_2(b) &= \prod_{i=1}^{M} \int_{\mathbb{Z}_2^2} \psi(\epsilon_i 2^{m_i} b y_{i1} y_{i2}) dy_{i1} dy_{i2} \\
J_3(b) &= \prod_{j=1}^{N} \int_{\mathbb{Z}_2^2} \psi(\epsilon_j 2^{n_j} b (z_{j1}^2 + z_{j1} z_{j2} + z_{j2}^2)) dz_{j1} dz_{j2}.
\end{aligned}$$

By Lemma 4.3, one has

$$J_1(\beta 2^{-k}) = \delta(k) (\frac{2}{\epsilon(k)}) 2^{\sum_{l_h < k-1} \frac{l_h - k + 1}{2}} \prod_{l_h < k-1} \psi(\frac{\epsilon_h \beta}{8}) (\frac{2}{\alpha})^{l(k-1,1)}.$$

By Lemma 4.4, one has

$$J_2(\beta 2^{-k}) = 2^{\sum_{m_i < k} m_i - k}$$

and

$$J_3(\beta 2^{-k}) = p(k) 2^{\sum_{n_j < k} n_j - k}.$$

Therefore

$$\begin{aligned}
&\alpha_2(\alpha 2^a, S) \\
&= 1 + \sum_{k=1}^{\infty} \delta(k) p(k) (\frac{2}{\epsilon(k)}) 2^{d(k)} \int_{\mathbb{Z}_2^*} (\frac{2}{\alpha})^{l(k-1,1)} \psi(\frac{-\mu_k(t)}{8} \beta) d\beta.
\end{aligned}$$

The last integral is always zero for $k > a + 3$ since $\mu_k(t) \notin \mathbb{Z}_2$ in this case.

When $l(k-1,1)$ is odd, the last integral is equal to $2^{-\frac{3}{2}} (\frac{2}{\mu_k(t)})$ by Lemma 4.3 and our notation (4.5). When $l(k-1,1)$ is even, the last integral is equal to

$$\begin{aligned}
\int_{\mathbb{Z}_2^*} \psi(\frac{-\mu_k(t)}{8} \beta) d\beta &= \frac{1}{2} \psi(-\frac{\mu_k(t)}{8}) \int_{\mathbb{Z}_2} \psi(\frac{-\mu_k(t)}{4} \beta) d\beta \\
&= \frac{1}{2} \psi(\frac{\mu_k(t)}{8}) \ \mathrm{char}(4\mathbb{Z}_2)(\mu).
\end{aligned}$$

Now Theorem 4.1 follows easily.

5. A proposition.

The purpose of this section is to prove a transformation formula needed for the main calculation in section 7. We fix an element $u \in \mathbb{Z}_p^*$ such that $(\frac{u}{p}) = -1$. Let $\mathcal{H} = \mathrm{Sym}_2(\mathbb{Q}_p) \cap \mathrm{GL}_2(\mathbb{Q}_p)$ be the set of nonsingular symmetric matrices over $\mathbb{Q}_p$, and let $\mathrm{GL}_2(\mathbb{Z}_p)$ acts on $\mathcal{H}$ via $g.X = {}^t g X g$. By a well-known theorem ([Ca, Theorem 8.3.1]), the following is a complete representative set of the $\mathrm{GL}_2(\mathbb{Z}_p)$-orbits in $\mathcal{H}$

$$p^b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad p^b \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}, \quad \text{and} \begin{pmatrix} u_1 p^{b_1} & 0 \\ 0 & u_2 p^{b_2} \end{pmatrix} \tag{5.1}$$

with $b_1 < b_2$ and $b \in \mathbb{Z}$ and $u_i = 1$ or u. Recall also that every $\mathrm{GL}_2(\mathbb{Z}_p)$-orbit is open in $\mathcal{H}$ since two quadratic forms over $\mathbb{Q}_p$ sufficiently close p-adically are $\mathrm{GL}_2(\mathbb{Z}_p)$-equivalent ([Ca, chapter 8]).

Proposition 5.1. *Let f be a locally constant bounded function on $Sym_2(\mathbb{Q}_p)$ such that $f \in L^1(Sym_2(\mathbb{Q}_p))$. Then*

$$\begin{aligned}
&\int_{Sym_2(\mathbb{Q}_p)} f(x)dx \\
&= \sum_{b \in \mathbb{Z}} p^{-3b} \left(\frac{1}{2(1-(\frac{-1}{p})p^{-1})} \int_{GL_2(\mathbb{Z}_p)} f(p^b\, {}^t x x)dx \right. \\
&\qquad \left. + \frac{1}{2(1+(\frac{-1}{p})p^{-1})} \int_{GL_2(\mathbb{Z}_p)} f(p^b\, {}^t x \; diag(1,u)x)dx \right) \\
&\quad + \frac{1}{4} \sum_{b_1<b_2, u_i=1 \text{ or } u} p^{-2b_1-b_2} \int_{GL_2(\mathbb{Z}_p)} f({}^t x \; diag(u_1p^{b_1}, u_2p^{b_2})x)dx
\end{aligned}$$

Proof. Since $\mathrm{Sym}_2(\mathbb{Q}_p) - \mathcal{H}$ is a closed subvariety of codimension 1, one has

$$\begin{aligned}
\int_{\mathrm{Sym}_2(\mathbb{Q}_p)} f(x)dx &= \int_{\mathcal{H}} f(x)dx \\
&= \sum \int_{\mathbb{O}} f(x)dx.
\end{aligned} \tag{5.2}$$

Here the sum runs over all $\mathrm{GL}_2(\mathbb{Z}_p)$-orbits $\mathbb{O}$ of $\mathcal{H}$. For each matrix $\mathrm{diag}(t_1, t_2)$, let $\mathbb{O}(t_1, t_2)$ be the $\mathrm{GL}_2(\mathbb{Z}_p)$-orbit of $\mathrm{diag}(t_1, t_2)$, and let $O(t_1, t_2)$ be the stabilizer of $\mathrm{diag}(t_1, t_2)$ in $\mathrm{GL}_2(\mathbb{Z}_p)$. Let dh be the Haar measure on $O(t_1, t_2)$ with total measure 1. Since the Haar measure on $\mathcal{H}$ is $\mathrm{GL}_2(\mathbb{Z}_p)$ invariant, there is a constant $C = C(t_1, t_2)$ such that for every locally constant function ϕ in $\mathrm{GL}_2(\mathbb{Z}_p)$, one has

$$C \int_{\mathrm{GL}_2(\mathbb{Z}_p)} f(x)dx = \int_{\mathbb{O}(t_1,t_2)} \int_{O(t_1,t_2)} f(hx)dhdx. \tag{5.3}$$

Take $\phi =$ char$(1 + p^n M_2(\mathbb{Z}_p))$ for sufficiently large integer n. Then the left hand side of (5.3) is Cp^{-4n}, and the right hand side is equal to

$$\operatorname{meas}(O(t_1,t_2) \cap (1 + p^n M_2(\mathbb{Z}_p))) \cdot \operatorname{meas}(\mathbb{O}_n(t_1,t_2)).$$

Here $\mathbb{O}_n(t_1,t_2)$ is the $(1 + p^n M_2(\mathbb{Z}_p))$-orbit of $\operatorname{diag}(t_1,t_2)$. Since

$$\begin{aligned} &{}^t(1 + p^n \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}) \operatorname{diag}(t_1,t_2)(1 + p^n \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}) \\ &= \begin{pmatrix} t_1 + 2t_1 p^n x_1 & t_1 p^n x_2 + t_2 p^n x_3 \\ t_1 p^n x_2 + t_2 p^n x_3 & t_2 + 2t_2 p^n x_4 \end{pmatrix} p^{2n} \begin{pmatrix} * & * \\ * & * \end{pmatrix}, \end{aligned}$$

one has

$$\operatorname{meas}(\mathbb{O}_n(t_1,t_2)) = p^{-3n-2b_1-b_2}.$$

On the other hand,

$$\operatorname{meas}(O(t_1,t_2) \cap (1 + p^n M_2(\mathbb{Z}_p))) = \frac{1}{[O(t_1,t_2) : O(t_1,t_2) \cap (1 + p^n M_2(\mathbb{Z}_p))]}.$$

Applying the lemma below, one has then

$$C(t_1,t_2) = p^{-2b_1-b_2} \mu(t_1,t_2) \tag{5.4}$$

where

$$\mu(t_1,t_2) = \begin{cases} \dfrac{1}{4} & \text{if } b_1 < b_2, \\ \dfrac{1}{2(1 - (\frac{-u_1u_2}{p})p^{-1})} & \text{if } b_1 = b_2. \end{cases} \tag{5.5}$$

$(t_i = u_i p^{b_i})$. Applying (5.3) – (5.5) to (5.2), one proves the proposition.

Lemma 5.2. *One has*

$$[O(t_1,t_2) : O(t_1,t_2) \cap (1 + p^n M_2(\mathbb{Z}_p))] = \mu(t_1,t_2)^{-1} p^n.$$

Proof. Direct computation gives

$$O(t_1,t_2) = SO(t_1,t_2) \propto \{\pm 1\}$$

where

$$\begin{aligned} SO(t_1,t_2) &= \{\begin{pmatrix} x_1 & x_2 \\ -\frac{t_1}{t_2} x_2 & x_1 \end{pmatrix} : x_i \in \mathbb{Z}_p, \frac{t_1}{t_2} x_2 \in \mathbb{Z}_p, x_1^2 + \frac{t_1}{t_2} x_2^2 = 1\} \\ &\cong \{(x_1,x_2) \in \mathbb{Z}_p^2 : x_2 \in \frac{t_2}{t_1}\mathbb{Z}_p, x_1^2 + \frac{t_1}{t_2} x_2^2 = 1\} \end{aligned}$$

and

$$O(t_1,t_2)\cap(1+p^nM_2(\mathbb{Z}_p))$$
$$\cong\{(x_1,x_2)\in\mathbb{Z}_p^2 : x_1\equiv 1 \bmod p^n, x_2\equiv 0 \bmod p^{n+b_2-b_1}, x_1^2+\frac{t_1}{t_2}x_2^2=1\}.$$

When $b_1<b_2$, the map $(x_1,x_2)\mapsto x_2$ is a two-to-one map from $SO(t_1,t_2)$ onto $p^{b_2-b_1}\mathbb{Z}_p$ and a one-to-one correspondence from $O(t_1,t_2)\cap(1+p^nM_2(\mathbb{Z}_p))$ onto $p^{n+b_2-b_1}\mathbb{Z}_p$. This proves the case $b_1<b_2$.

When $b_1=b_2$, $\frac{t_1}{t_2}=\frac{u_1}{u_2}\in\mathbb{Z}_p^*$. Let $\alpha=\sqrt{-\frac{u_1}{u_2}}$. If $\alpha\in\mathbb{Z}_p^*$, i.e., $(\frac{-u_1u_2}{p})=1$, then $(x_1,x_2)\mapsto x_1+\alpha x_2$ is an isomorphism between $SO(t_1,t_2)$ to $\mathbb{Z}_p^*$ and maps $O(t_1,t_2)\cap(1+p^nM_2(\mathbb{Z}_p))$ onto $1+p^n\mathbb{Z}_p$. This implies

$$[SO(t_1,t_2):O(t_1,t_2)\cap(1+p^nM_2(\mathbb{Z}_p))]=[\mathbb{Z}_p^*:1+p^n\mathbb{Z}_p]=p^n(1-p^{-1}).$$

If $\alpha\notin\mathbb{Z}_p^*$, i.e., $(\frac{-u_1u_2}{p})=-1$, then $K=\mathbb{Q}_p(\alpha)$ is a quadratic extension. Let K^1 be the norm-1 subgroup of K and let

$$K^1(n)=\{z\in K^1 : z\equiv 1 \bmod p^n\}.$$

Then the map $(x_1,x_2)\mapsto x_1+\alpha x_2$ gives an isomorphism from $SO(t_1,t_2)$ onto K^1 and from $O(t_1,t_2)\cap(1+p^nM_2(\mathbb{Z}_p))$ onto $K^1(n)$. So in this case, one has

$$[SO(t_1,t_2):O(t_1,t_2)\cap(1+p^nM_2(\mathbb{Z}_p))]=[K^1:K^1(n)]=p^n(1+p^{-1}).$$

6. Gauss integrals over GL_2.

For any matrix $M=\begin{pmatrix}t_1 & t_2\\ t_3 & t_4\end{pmatrix}\in M_2(\mathbb{Q}_p)$, we compute the following Gauss integral in this section.

$$I^*(t_1,t_2,t_3,t_4)=I(M)=\int_{\mathrm{GL}_2(\mathbb{Z}_p)}\psi(\sum t_ix_i^2)dx_1dx_2dx_3dx_4. \tag{6.1}$$

Here dx_i is the Haar measure on $\mathbb{Q}_p$ so that $\mathrm{meas}(\mathbb{Z}_p)=1$. The integral is a generalization of the integral (2.2) to GL_2, and will be needed in next section. We will use either of the two notations whenever convenient. The following lemma is obvious from the definition.

Lemma 6.1. (1) *The value of the integral $I^*(t_1,t_2,t_3,t_4)$ does not change under any of the following operations: switching t_1 with t_4, switching t_2 with t_3, or switching the pair (t_1,t_4) with the pair (t_2,t_3).*

(2) *For any $\alpha\in\mathbb{Z}_p^*$, one has*

$$I(\alpha^2t_1,\alpha^2t_2,t_3,t_4)=I(t_1,t_2,t_3,t_4).$$

In other words, multiplying a column or a row of M by a square does not change the value of $I(M)$.

Lemma 6.2. (1)

$$\begin{aligned}
&I^*(t_1,t_2,t_3,t_4)\\
&= I^*(t_1)I^*(t_4)I^*(t_2)I(t_3) + I^*(t_1)I(t_4)I^*(t_2)I^*(t_3) + p^{-1}I^*(t_1)I^*(t_4)I(p^2t_2)I(t_3)\\
&+ p^{-1}I(p^2t_1)I(t_4)I^*(t_2)I^*(t_3) - I^*(t_1)I^*(t_4)I^*(t_2)I^*(t_3)\\
&- p^{-4}\ char((p^{-1}\mathbb{Z}_p)^4)(t_1,t_2,t_3,t_4)\sum_{x_i\in\mathbb{F}_p^*, x_1x_4=x_2x_3}\psi(\sum t_1x_i^2).
\end{aligned}$$

Here $\mathbb{F}_p$ is the finite field of p elements, $I(t)$ and $I^*(t)$ are integrals given by (2.1) *and* (2.2).

(2) *For $\alpha_i \in \mathbb{Z}_p$, one has*

$$I^*(\alpha_1p^{-1},\alpha_2p^{-1},\alpha_3p^{-1},\alpha_4p^{-1}) = p^{-4}\sum_{X\in\ GL_2(\mathbb{F}_p)} e_p(-\sum_{i=1}^4(\alpha_ix_i^2)).$$

Here $e_p(x) = e^{\frac{2\pi i x}{p}}$, and $X = \begin{pmatrix} x_1 & x_2\\ x_3 & x_4\end{pmatrix}$.

Proof. By the definition of ψ, one has $\psi(xp^{-1}) = e_p(-x)$ for $x \in \mathbb{Z}_p$. Now (2) is obvious. To prove (1), first notice that $\mathrm{GL}_2(\mathbb{Z}_p) \subset \mathbb{Z}_p^4$ is the disjoint union of the subsets (coordinates are in the order (x_1,x_4,x_2,x_3))

$$p\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p^*\times\mathbb{Z}_p^*,\quad \mathbb{Z}_p^*\times p\mathbb{Z}_p\times\mathbb{Z}_p^*\times\mathbb{Z}_p^*,\quad \mathbb{Z}_p^*\times\mathbb{Z}_p^*\times p\mathbb{Z}_p\times\mathbb{Z}_p,$$

and

$$\{(x_1,x_4,x_2,x_3)\in\mathbb{Z}_p^{*3}\times\mathbb{Z}_p : x_3\not\equiv x_2^{-1}x_1x_4 \bmod p\}.$$

So

$$\begin{aligned}
&I^*(t_1,t_2,t_3,t_4)\\
&= p^{-1}I(p^2t_1)I(t_4)I^*(t_2)I^*(t_3) + p^{-1}I^*(t_1)I(p^2t_4)I^*(t_2)I^*(t_3)\\
&+ p^{-1}I^*(t_1)I^*(t_4)I(p^2t_2)I(t_3) + I^*(t_1)I^*(t_4)I^*(t_2)I(t_3)\\
&- \int_{\mathbb{Z}_p^{*3}}\psi(t_1x_1^2+t_2x_2^2+t_4x_4^2)dx_1dx_2d_4\int_{x_3\equiv x_2^{-1}x_1x_4 \bmod p}\psi(t_3x_3^2)dx_3.
\end{aligned}$$

Since $p^{-1}I(p^2t_4) = I(t_4) - I^*(t_4)$, it suffices to show that

$$\begin{aligned}
&\int_{\mathbb{Z}_p^{*3}}\psi(t_1x_1^2+t_2x_2^2+t_4x_4^2)dx_1dx_2d_4\int_{x_3\equiv x_2^{-1}x_1x_4 \bmod p}\psi(t_3x_3^2)dx_3\\
&= p^{-4}\ \mathrm{char}((p^{-1}\mathbb{Z}_p)^4)(t_1,t_2,t_3,t_4)\sum_{x_i\in\mathbb{F}_p^*, x_1x_4=x_2x_3}\psi(\sum t_1x_i^2).
\end{aligned}\tag{6.2}$$

By Lemma 2.1, the left hand side of (6.2) is

$$p^{-1}\ \mathrm{char}(p^{-1}\mathbb{Z}_p)(t_3)\int_{\mathbb{Z}_p^{*3}}\psi(t_1x_1^2)\psi(t_2x_2^2)\psi(t_4x_4^2)\psi(t_3x_2^{-2}x_1^2x_4^2)dx_1dx_2dx_4.$$

This is zero unless $t_1, t_4 \in p^{-1}\mathbb{Z}_p$ since it contains a factor $I^*(t_1+t_3x_2^{-1}x_2^2)$ resp. $I^*(t_4+t_3x_2^{-2}x_1^2)$. The same proof as in Lemma 2.1(3) would make the integral zero unless $t_2 \in p^{-1}\mathbb{Z}_p$ (under the condition $t_3 \in p^{-1}\mathbb{Z}_p$). On the other hand, if $t_1, ..., t_4 \in p^{-1}\mathbb{Z}_p$, then the integral equals

$$\begin{aligned} &p^{-1}\sum_{x_1,x_2,x_4\in\mathbb{F}_p^*}\psi(t_1x_1^2)\psi(t_2x_2^2)\psi(t_4x_4^2)\psi(t_3x_2^{-2}x_1^2x_4^2)\int_{(1+p\mathbb{Z}_p)^3}dy\\ &= p^{-4}\sum_{x_i\in\mathbb{F}_p^*, x_1x_4=x_2x_3}\psi(\sum t_ix_i^2).\end{aligned}$$

This finishes the proof of Lemma 6.2.

To obtain a more precise formula for $I^*(t_1, t_2, t_3, t_4)$, write $t_i = \alpha_ip^{a_i}$ with $\alpha_i \in \mathbb{Z}_p^*$ and $a_i \in \mathbb{Z}$. We may and will assume that $a_1 \le a_2, a_4$, and $a_2 \le a_3$ without loss of generality in the rest of this section (see Lemma 6.1).

Corollary 6.3. *With notation and assumption as above, one has*

$$I^*(t_1, t_2, t_3, t_4) = \begin{cases} (1-p^{-1})^2(1+p^{-1}) & \text{if } a_1 \ge 0,\\ I(t_1)I(t_4)I^*(t_2)I^*(t_3) & \text{if } a_1 \le -2.\end{cases}$$

In particular, if $a_1, a_2 \le -2$, then $I^*(t_1, t_2, t_3, t_4) = 0$.

Proof. The case $a_1 \ge 0$ is trivial. If $a_1 \le -2$ then $I^*(t_1) = 0$ by Lemma 2.1. So Lemma 6.2 implies

$$I^*(t_1, t_2, t_3, t_4) = p^{-1}I(p^2t_1)I(t_4)I^*(t_2)I^*(t_3) = I(t_1)I(t_4)I^*(t_2)I^*(t_3).$$

The case $a_1 = -1$ is a little bit more complicated and more interesting. We first assume at least one of a_i, say a_4 is nonnegative. Then (assuming $a_4 \ge 0$)

$$\begin{aligned} &\sum_{x_i\in\mathbb{F}_p^*, x_1x_4=x_2x_3}\psi(\sum t_ix_i^2)\\ &= (\sum_{x_1\in\mathbb{F}_p^*}\psi(t_1x_1^2))(\sum_{x_2\in\mathbb{F}_p^*}\psi(t_2x_2^2))(\sum_{x_3\in\mathbb{F}_p^*}\psi(t_3x_3^2))\\ &= p^3I^*(t_1)I^*(t_2)I^*(t_3)\end{aligned}$$

Also notice that $I(t_4) = 1$ and $I^*(t_4) = 1 - p^{-1}$ in this case. So Lemmas 6.2 and 2.1 imply

Lemma 6.4. *Let the notation be as above. Assume that $a_1 = -1$ and $a_4 \geq 0$. Then*

$$I^*(t_1, t_2, t_3, t_4) = (1 - p^{-1}) I^*(t_1) I(t_2) I(t_3) + p^{-1} I^*(t_2) I^*(t_3).$$

More precisely, one has

$$\begin{aligned} &I^*(t_1, t_2, t_3, t_4) \\ &= \begin{cases} (1-p^{-1})\left(\frac{\delta_p}{\sqrt{p}}(\frac{-\alpha_1}{p}) - \frac{1}{p^2}\right) & \text{if } a_2, a_3 \geq 0, \\ p^{-1}(1-p^{-1})\left((\frac{-\alpha_1\alpha_2}{p}) - \frac{1}{p}\right) & \text{if } a_2 = -1, a_3 \geq 0, \\ p^{-3}(1 + (\frac{-\alpha_2\alpha_3}{p})) + \frac{\delta_p}{p\sqrt{p}}(\frac{\alpha_1\alpha_2\alpha_3}{p}) & \\ \quad - \frac{\delta_p}{p^2\sqrt{p}}\left((\frac{\alpha_1\alpha_2\alpha_3}{p}) + (\frac{-\alpha_2}{p}) + (\frac{-\alpha_3}{p})\right) & \text{if } a_2 = a_3 = -1. \end{cases} \end{aligned}$$

Finally, it comes to the case where $a_1 = a_2 = a_3 = a_4 = -1$. By Lemma 6.2, it reduces to compute the following Gauss sum

$$S(M) = S(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sum_{X \in \mathrm{GL}_2(\mathbb{F}_p)} e_p(\sum \alpha_i x_i^2) \tag{6.3}$$

where $M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in M_2(\mathbb{F}_p)$, $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, and $e_p(x) = e^{\frac{2\pi i x}{p}}$. One has

$$I^*(\alpha_1 p^{-1}, \alpha_2 p^{-1}, \alpha_3 p^{-1}, \alpha_4 p^{-1}) = p^{-4} S(-\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4) \tag{6.4}$$

for $\alpha_i \in \mathbb{Z}_p$. Here we also use α_i for its image in $\mathbb{F}_p$. When some of $\alpha_i = 0$, the sum (or the integral) is given by Lemma 6.4. The following lemma was proved with the help of Don Zagier. I thank him for his help and for allowing me to publish it here.

Lemma 6.5. *Let $M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$ with $\alpha_i \in \mathbb{F}_p^*$.*

(1) *When $\det M = 0$, i.e, $M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta\alpha_1 & \beta\alpha_2 \end{pmatrix}$ with some $\beta \in \mathbb{F}_p^*$, one has*

$$S(M) = -p^2((\frac{-\beta}{p}) + (\frac{-\alpha_1\alpha_2}{p}))(1 - (\frac{-\alpha_1\alpha_2}{p})p^{-1}).$$

(2) *When $\det M \neq 0$, let $\lambda = \frac{\alpha_2\alpha_3}{\alpha_1\alpha_4}$, and let*

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

be the corresponding elliptic curve over $\mathbb{F}_p$. Let

$$a_p(\lambda) = a_p(E_\lambda) = \#E_\lambda(\mathbb{F}_p) - p - 1.$$

be the trace of the Frobenious on E_λ. Then

$$\begin{aligned} S(M) = (\frac{\prod \alpha_i}{p})p^2 - (\frac{\alpha_1\alpha_4}{p})pa_p(\lambda) + p \\ - p\sqrt{p}\delta_p(\frac{\det M}{p})\left((\frac{\alpha_1}{p}) + (\frac{-\alpha_2}{p}) + (\frac{-\alpha_3}{p}) + (\frac{\alpha_4}{p})\right).\end{aligned}$$

Here δ_p is the number defined by (1.9)*, i.e., it is* 1 *or i depending on whether $p \equiv 1 \mod 4$ or $p \equiv -1 \mod 4$.*

Proof. By the classical Gauss sum formula, one has

$$\begin{aligned} S(M) &= \sum_{X \in M_2(\mathbb{F}_p)} e_p(\sum \alpha_i x_i^2) - \sum_{\det X = 0} e_p(\sum \alpha_i x_i^2) \\ &= (\frac{\prod \alpha_i}{p})p^2 - \sum_{\det X=0} e_p(\sum \alpha_i x_i^2) \end{aligned}$$

Since $\det X = 0$ means that either $X = \begin{pmatrix} a & b \\ \mu a & \mu b \end{pmatrix}$ with $a, b, \mu \in \mathbb{F}_p$ or $X = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ with $X = 0$ being counted $p+1$ times, one has

$$\begin{aligned} \sum_{\det X=0} e_p(\sum \alpha_i x_i^2) = \sum_\mu \sum_a e_p((\alpha_1 + \alpha_3\mu^2)a^2) \sum_b e_p((\alpha_2 + \alpha_4 \mu^2)b^2) \\ + \sum_{c,d} e_p(\alpha_3 c^2 + \alpha_4 d^2) - p. \end{aligned}$$

Applying the classical Gauss sum formula to the second term, one obtains

$$\begin{aligned} S(M) = (\frac{\prod \alpha_i}{p})p^2 + p - p(\frac{-\alpha_3\alpha_4}{p}) \\ - \sum_\mu \sum_a e_p((\alpha_1 + \alpha_3\mu^2)a^2) \sum_b e_p((\alpha_2 + \alpha_4\mu^2)b^2) \end{aligned} \tag{6.5}$$

Notice that
(6.6)

$$\sum_a e_p((\alpha_1 + \alpha_3 \mu^2)a^2) = \begin{cases} p & \text{if } (\frac{-\alpha_1\alpha_3}{p}) = 1 \& \mu = \pm\sqrt{-\frac{\alpha_1}{\alpha_3}}, \\ \sqrt{p}\delta_p(\frac{\alpha_1 + \alpha_3\mu^2}{p}) & \text{otherwise.} \end{cases}$$

Case 1 When $\det M = 0$, $\frac{-\alpha_1}{\alpha_3} = \frac{-\alpha_2}{\alpha_4}$. In this case, the roots $\mu = \pm\sqrt{\frac{-\alpha_1}{\alpha_3}}$ and $\mu = \pm\sqrt{\frac{-\alpha_2}{\alpha_4}}$ are the same if they exist in $\mathbb{F}_p$. Let $\beta = \frac{\alpha_3}{\alpha_1}$, then one has by (6.6)

$$\begin{aligned}
&\sum_\mu \sum_a e_p((\alpha_1+\alpha_3\mu^2)a^2) \sum_b e_p((\alpha_2+\alpha_4\mu^2)b^2) \\
&= \sum_\mu p(\frac{-1}{p})(\frac{\alpha_1+\alpha_3\mu^2}{p})(\frac{\alpha_2+\alpha_4\mu^2}{p}) + (1+(\frac{-\beta}{p}))p^2 \\
&= (\frac{-\alpha_1\alpha_2}{p})p\sum_\mu(\frac{1+\beta\mu^2}{p})^2 + (1+(\frac{-\beta}{p}))p^2 \\
&= (\frac{-\alpha_1\alpha_2}{p})p(p-1-(\frac{-\beta}{p})) + (1+(\frac{-\beta}{p}))p^2 \\
&= (1+(\frac{-\beta}{p})+(\frac{-\alpha_1\alpha_2}{p}))p^2 - (\frac{-\alpha_1\alpha_2}{p})(1+(\frac{-\beta}{p}))p.
\end{aligned}$$

Plugging this into (6.5), one proves (1).

Case 2 When $\det M \neq 0$, the roots $\mu = \pm\sqrt{\frac{-\alpha_1}{\alpha_3}}$ and $\mu = \pm\sqrt{\frac{-\alpha_2}{\alpha_4}}$ are different if they exist in $\mathbb{F}_p$. So

$$\begin{aligned}
&\sum_\mu \sum_a e_p((\alpha_1+\alpha_3\mu^2)a^2) \sum_b e_p((\alpha_2+\alpha_4\mu^2)b^2) \\
&= \sum_\mu p(\frac{-1}{p})(\frac{\alpha_1+\alpha_3\mu^2}{p})(\frac{\alpha_2+\alpha_4\mu^2}{p}) \\
&\quad + (1+(\frac{-\alpha_1\alpha_3}{p}))p\sum_b e_p((\alpha_2-\frac{\alpha_1\alpha_4}{\alpha_3})b^2) \\
&\quad + (1+(\frac{-\alpha_2\alpha_4}{p}))p\sum_a e_p((\alpha_1-\frac{\alpha_2\alpha_3}{\alpha_4})a^2) \\
&= (\frac{-\alpha_3\alpha_4}{p})p\sum_{x\in\mathbb{F}_p}(\frac{x+\alpha_1\alpha_3^{-1}}{p})(\frac{x+\alpha_2\alpha_4^{-1}}{p})(1+(\frac{x}{p})) \\
&\quad + ((\frac{\alpha_1}{p})+(\frac{-\alpha_3}{p}))(\frac{\det M}{p})p\sqrt{p}\delta_p + ((\frac{-\alpha_2}{p})+(\frac{\alpha_4}{p}))(\frac{\det M}{p})p\sqrt{p}\delta_p.
\end{aligned}$$

So

$$\sum_\mu \sum_a e_p((\alpha_1+\alpha_3\mu^2)a^2)\sum_b e_p((\alpha_2+\alpha_4\mu^2)b^2)$$
$$=(\frac{-\alpha_3\alpha_4}{p})p\sum_x(\frac{(x+\alpha_1\alpha_3^{-1})(x+\alpha_2\alpha_4^{-1})}{p})$$
$$+(\frac{-\alpha_3\alpha_4}{p})p\sum_x(\frac{x(x+\alpha_1\alpha_3^{-1})(x+\alpha_2\alpha_4^{-1})}{p})$$
$$+p\sqrt{p}\delta_p(\frac{\det M}{p})((\frac{\alpha_1}{p})+(\frac{-\alpha_2}{p})+(\frac{-\alpha_3}{p})+(\frac{\alpha_4}{p})).$$

Given a polynomial $f(x)\in\mathbb{F}_p[x]$, let C_f^a be the affine curve defined by $y^2=f(x)$. Since, for a fixed $x\in\mathbb{F}_p$, the equation $y^2=f(x)$ has $1+(\frac{f(x)}{p})$ solutions in $\mathbb{F}_p$, one has

$$\sum_{x\in\mathbb{F}_p}(\frac{f(x)}{p})=\#C_f^a(\mathbb{F}_p)-p. \tag{6.7}$$

It is easy to see from this that

$$\sum_{x\in\mathbb{F}_p}(\frac{x^2+ax+b}{p})=\begin{cases}p-1 & \text{if } a^2-4b=0,\\ -1 & \text{if } a^2-4b\neq 0.\end{cases} \tag{6.8}$$

In particular, one has ($\det M\neq 0$)

$$\sum_x(\frac{(x+\alpha_1\alpha_3^{-1})(x+\alpha_2\alpha_4^{-1})}{p})=-1.$$

Next, let E_M be the elliptic curve defined by

$$E_M: y^2=x(x+\alpha_1\alpha_3^{-1})(x+\alpha_2\alpha_4^{-1}).$$

Notice that E_M has one extra point (infinity) to its affine points. So (6.7) gives

$$\sum_x(\frac{x(x+\alpha_1\alpha_3^{-1})(x+\alpha_2\alpha_4^{-1})}{p})=a_p(E_M).$$

So

$$\sum_\mu \sum_a e_p((\alpha_1+\alpha_3\mu^2)a^2)\sum_b e_p((\alpha_2+\alpha_4\mu^2)b^2)$$
$$=-(\frac{-\alpha_3\alpha_4}{p})p+(\frac{-\alpha_3\alpha_4}{p})pa_p(E_M)$$
$$+p\sqrt{p}\delta_p(\frac{\det M}{p})((\frac{\alpha_1}{p})+(\frac{-\alpha_2}{p})+(\frac{-\alpha_3}{p})+(\frac{\alpha_4}{p})).$$

Plugging this into (6.5) and comparing the result with the desired formula in (2) leads us to verify

$$a_p(E_M) = (\frac{-\alpha_1\alpha_3}{p})a_p(\lambda). \tag{6.9}$$

Notice that the map $x \mapsto -\alpha_1\alpha_3^{-1}x$ and $y \mapsto (-\alpha_1\alpha_3^{-1})^2 y$ gives a bijection between $E'(\mathbb{F}_p)$ and $E_M(\mathbb{F}_p)$, where

$$E' : -\alpha_1\alpha_3^{-1}y^2 = x(x-1)(x-\lambda)$$

is the quadratic twist of E_λ by $-\alpha_1\alpha_3^{-1}$ and $\lambda = \frac{\alpha_2\alpha_3}{\alpha_1\alpha_4}$. So (6.9) follows from the general fact that

$$a_p(E^d) = (\frac{d}{p})a_p(E) \tag{6.10}$$

for any elliptic curve E over $\mathbb{F}_p$ and any quadratic twist E^d with $d \in \mathbb{F}_p^*$.

Corollary 6.6. (1) *For $\alpha_i, \beta \in \mathbb{Z}_p^*$, one has*

$$\begin{aligned} &I^*(\alpha_1 p^{-1}, \alpha_2 p^{-1}, \alpha_1\beta p^{-1}, \alpha_2\beta p^{-1}) \\ &= -p^{-2}((\frac{-\beta}{p}) + (\frac{-\alpha_1\alpha_2}{p}))(1 - (\frac{-\alpha_1\alpha_2}{p})p^{-1}). \end{aligned}$$

(2) *For $\alpha_i \in \mathbb{Z}_p^*$ with $\alpha_1\alpha_4 \neq \alpha_2\alpha_3$, one has*

$$\begin{aligned} &I^*(\alpha_1 p^{-1}, \alpha_2 p^{-1}, \alpha_3 p^{-1}, \alpha_4 p^{-1}) \\ &= (\frac{\prod \alpha_i}{p})p^{-2} + p^{-3} - (\frac{\alpha_1\alpha_4}{p})p^{-3}a_p(\lambda) \\ &\quad - p^{-3}\sqrt{p}\delta_p(\frac{\alpha_1\alpha_4 - \alpha_2\alpha_3}{p})\left((\frac{-\alpha_1}{p}) + (\frac{\alpha_2}{p}) + (\frac{\alpha_3}{p}) + (\frac{-\alpha_4}{p})\right). \end{aligned}$$

7. The case $n = 2 \neq p$.

Let the notation be as in section 1 and assume $p \neq 2$. We further set

$$\delta^\pm(k) = \frac{1 \pm (-1)^{l(k,1)}}{2}, \tag{7.1}$$

$$f_1(\alpha p^a) = \begin{cases} -\frac{1}{p} & \text{if } l(a+1,1) \text{ is even,} \\ (\frac{\alpha}{p})\frac{1}{\sqrt{p}} & \text{if } l(a+1,1) \text{ is odd,} \end{cases} \tag{7.2}$$

and

$$f_2(\alpha p^a) = \begin{cases} -\frac{1}{p} & \text{if } l(a+1,1) \text{ is odd,} \\ (\frac{-\alpha}{p})\frac{1}{\sqrt{p}} & \text{if } l(a+1,1) \text{ is even.} \end{cases} \tag{7.3}$$

Finally, set

$$g(k) = \begin{cases} \delta^+(k) & \text{if } a-k \text{ is even,} \\ (\frac{\alpha_1}{p})\delta^-(k) & \text{if } a-k \text{ is odd.} \end{cases} \tag{7.4}$$

We remark that all those terms are independent of l or r. To make the complicated formula manageable and to give it some structure, we group the terms of the formula into 2 groups (R_1 and R_2) and 12 polynomials $I_{i,j}$. The definitions depend on whether $a = b$ or not. The case $a = b$ is much easier and can be served as a check when one gets tired of the long calculation.

When $a = b$, we define $I_{i,j} = I_{i,j}(X, T, S)$ as follows :

$$I_{1,1} = (1-p^{-2}) \sum_{0<k\le a} v(k)\delta^+(k)p^{k+d(k)}X^k,$$

$$I_{1,2} = \delta^+(1+a)v(1+a)((\frac{-\alpha_1\alpha_2}{p}) - p^{-1})p^{a+d(a+1)}X^{a+1},$$

$$I_{1,3} = I_{1,4} = 0,$$

and

$$I_{2,1} = (1-p^{-2}) \sum_{0<k_2<k_1\le a} v(k_1)v(k_2)\delta^+(k_1)\delta^+(k_2)p^{k_1+d(k_1)+d(k_2)}X^{k_1+k_2}$$

$$= \sum_{0<k<2a} C_k X^k,$$

$$I_{2,2} = v(a+1)\delta^+(a+1)((\frac{-\alpha_1\alpha_2}{p}) - p^{-1}) \sum_{0<k\le a} \delta^+(k)v(k)p^{a+d(a+1)+d(k)}X^{a+1+k},$$

$$I_{2,6} = -((\frac{-\alpha_1\alpha_2}{p})\delta^+(1+a) + \delta^-(1+a))p^{a-1+2d(1+a)}X^{2a+2}$$

$$I_{2,8} = \sum_{0<k\le a} (\delta^+(k) + p^{-1}\delta^-(k))p^{k+2d(k)}X^{2k},$$

$$I_{2,3} = I_{2,4} = I_{2,5} = I_{2,7} = 0.$$

Here

$$C_k = (1-p^{-2}) \sum_{\substack{0<k_2<k_1\le a \\ k_1+k_2=k \\ l(k_i,1) \text{ even}}} v(k_1)v(k_2)p^{k_1+d(k_1)+d(k_2)}. \tag{7.5}$$

When $a < b$, we define

$$I_{1,1} = (1 - p^{-2}) \sum_{0<k\le a} v(k)\delta^+(k)p^{k+d(k)}X^k,$$

$$I_{1,2} = v(a+1)\left((\frac{\alpha_1}{p})\frac{1}{\sqrt{p}}\delta^-(a+1) - p^{-2}\delta^+(a+1)\right) p^{a+1+d(a+1)}X^{a+1},$$

$$I_{1,3} = (1 - p^{-1}) \sum_{a+1<k<b+1} v(k)g(k)p^{\frac{a+k}{2}+d(k)}X^k,$$

$$I_{1,4} = v(b+1)p^{\frac{a+b+1}{2}+d(b+1)}X^{b+1} \cdot \begin{cases} f_1(\alpha_2 p^b) & \text{if } a \not\equiv b \bmod 2, \\ (\frac{\alpha_1}{p})f_2(\alpha_2 p^b) & \text{if } a \equiv b \bmod 2, \end{cases}$$

and

$$\begin{aligned} I_{2,1} &= (1 - p^{-2}) \sum_{0<k_2<k_1\le a} v(k_1)v(k_2)\delta^+(k_1)\delta^+(k_2)p^{k_1+d(k_1)+d(k_2)}X^{k_1+k_2} \\ &= \sum_{0<k<2a} C_k X^k, \end{aligned}$$

$$\begin{aligned} I_{2,2} = &\left((\frac{\alpha_1}{p})\frac{1}{\sqrt{p}}\delta^-(a+1) - p^{-2}\delta^+(a+1)\right) \\ &\cdot \sum_{0<k\le a} v(a+1)v(k)\delta^+(k)p^{a+1+d(a+1)+d(k)}X^{a+1+k}, \end{aligned}$$

$$I_{2,3} = (1 - p^{-1}) \sum_{0<k_2<a+1<k_1\le b} v(k_1)v(k_2)\delta^+(k_2)g(k_1)p^{\frac{a+k_1}{2}+d(k_1)+d(k_2)}X^{k_1+k_2}.$$

We also define

$$\begin{aligned} I_{2,4} = &\sum_{0<k\le a} v(b+1)v(k)\delta^+(k)p^{\frac{a+b+1}{2}+d(b+1)+d(k)}X^{b+1+k} \\ &\cdot \begin{cases} f_1(\alpha_2 p^b) & \text{if } a \not\equiv b \bmod 2, \\ (\frac{\alpha_1}{p})f_2(\alpha_2 p^b) & \text{if } a \equiv b \bmod 2, \end{cases} \end{aligned}$$

$$I_{2,5} = v(a+1)f_1(\alpha_1 p^a) \sum_{a+1<k<b+1} v(k)g(k)p^{\frac{a+k_1}{2}+d(a+1)+d(k)}X^{a+1+k},$$

$$\begin{aligned} I_{2,6} = &v(a+1)v(b+1)f_1(\alpha_1 p^a)p^{\frac{a+b+1}{2}+d(a+1)+d(b+1)}X^{a+b+2} \\ &\cdot \begin{cases} f_1(\alpha_2 p^b) & \text{if } a \not\equiv b \bmod 2, \\ (\frac{\alpha_1}{p})f_2(\alpha_2 p^b) & \text{if } a \equiv b \bmod 2, \end{cases} \end{aligned}$$

and

$$\begin{aligned} I_{2,7} &= \delta^-(a+1)p^{a+2d(a+1)}X^{2a+2}, \\ I_{2,8} &= \sum_{0<k\le a} (\delta^+(k) + p^{-1}\delta^-(k))p^{k+2d(k)}X^{2k}. \end{aligned}$$

Now we are ready to state the main formula.

Theorem 7.1. *Set $X = p^{-r}$. Then*

$$\alpha(X, T^l, S^l) = R_0(X) + p^{2l}X^l R_1(X) + p^{3l}X^{2l}R_2(X) + l\beta^l(X), \tag{7.6}$$

where $R_i(X) = R_i(X, T, S)$ are given by

$$R_0 = 1, \quad R_1 = \sum_{i=1}^{4} I_{1,i}, \quad R_2 = (1-p^{-1})\sum_{i=1}^{8} I_{2,i} + p^{-1}I_{2,6},$$

and

$$\beta^l(X) = (1-p^{-2})p^{2l}X^l + (1-p^{-1})p^{3l}X^{2l}(1+R_1(X)).$$

Corollary 7.2. *Let the notation be as in Theorem* 7.1. *Then*

$$\alpha_p(X, T^1, S^1) - p^3X^2\alpha(X, T, S) = (1-X)(1+p^2X+p^2XR_1(X)).$$

Moreover, if $\alpha(1, T, S) = \alpha_p(T, S_0) = 0$, then

$$\alpha'(1, T^1, S^1) = p^3\alpha'(1, T, S) - (1+p^2+p^2R_1(1)).$$

Here α' is the derivative of $\alpha(X, T^l, S^l)$ with respect to X.

Remark 7.3. When S is unimodular, one can verify (by tedious calculation) that our formula for $X = 1$ and $l = 0$ coincide with those of Kitaoka's ([Ki, Theorem 2]).

The rest of this section is devoted to the proof of Theorem 7.1. It is quite technical and can be skipped. We first do an easy reduction.

Lemma 7.4. *If Theorem* 7.1 *is true for $l = r = 0$ (all S and T), then it is true in general.*

Proof. (sketch) Assume that Theorem 7.1 is true for $l = r = 0$. Since $\alpha_p(T^1, S^1) = p^3\alpha_p(T, S)$, it is also true for $l = 1$ and $r = 0$. In general, denote $L_r^l(k, 1)$ and so on for the data corresponding to S_r^l instead of S. Then for any $k \ge 0$, one has

$$\begin{aligned} l_r^l(k+l, 1) &\equiv l(k, 1) \bmod 2, \\ \delta_r^{l,\pm}(k+l) &= \delta^{\pm}(k), \\ v_r^l(k+l) &= v(k), \end{aligned}$$

and

$$d_r^l(k+l) = d(k) - rk + l - rl.$$

So the associated functions f_i and g (to T^l and S_r^l) are independent of l or r. Now applying the formula just proved to $T=T^l$ and $S=S_r^l$, one finds

$$\alpha_p(T^l,S_r^l)=1+p^{2l}X^l\sum_{i=1}^{4}I_{1,i}^l+(1-p^{-1})p^{3l}X^{2l}\sum_{i=1}^{8}I_{2,i}^l+p^{-1}p^{3l}X^{2l}I_{2,6},$$

where $I_{i,j}^l$ are almost the same as $I_{i,j}$ except for that the summation is over $l<k+l\le a+l$, for example, instead of $0<k\le a$. Notice that $d(0)=0$, $v(0)=1$, and $0\le l\le 1$. So

$$\begin{aligned}I_1^l&=(1-p^{-2})\sum_{\substack{0<k+l\le a+l\\ l(k,1)\text{ even}}}v(k)p^{k+d(k)}X^k\\&=(1-p^{-2})\sum_{\substack{-l<k\le a\\ l(k,1)\text{ even}}}v(k)p^{k+d(k)}X^k\\&=l(1-p^{-2})+I_1.\end{aligned}$$

For the same reason, one has $I_{1,i}^l=I_{1,i}$ for $i\ne 1$, and

$$I_{2,j}^l=\begin{cases}lI_{1,j}+I_{2,j} & \text{if } 1\le j\le 4,\\ I_{2,j} & \text{if } 5\le j\le 7,\\ l+I_{2,j} & \text{if } j=8.\end{cases}$$

Combining the formulae, one finds that theorem 7.1 is true in general.

Lemma 7.5. *Let $J(p^{-k})$ be defined by* (2.3). *Then*

$$\alpha_p(T,S)=\sum_{k\in\mathbb{Z}}p^{3k}J(p^{-k})^2R(k)+\sum_{k_1>k_2}p^{2k_1+k_2}J(p^{-k_1})J(p^{-k_2})R(k_1,k_2). \tag{7.7}$$

Here

$$\begin{aligned}R(k)=&\frac{1}{2(1-(\frac{-1}{p})p^{-1})}I^*(-\alpha_1p^{a-k},-\alpha_1p^{a-k},-\alpha_2p^{b-k},-\alpha_2p^{b-k})\\&+\frac{(-1)^{l(k,1)}}{2(1+(\frac{-1}{p})p^{-1})}I^*(-\alpha_1p^{a-k},-\alpha_1up^{a-k},-\alpha_2p^{b-k},-\alpha_2up^{b-k}),\end{aligned}$$

and

$$\begin{aligned}R(k_1,k_2)=&\frac{1}{4}\sum_{\beta_1,\beta_2=1\ or\ u}(\frac{\beta_1}{p})^{l(k_1,1)}(\frac{\beta_2}{p})^{l(k_2,1)}\\&\cdot I^*(-\alpha_1\beta_1p^{a-k_1},-\alpha_1\beta_2p^{a-k_2},-\alpha_2\beta_1p^{b-k_1},-\alpha_2\beta_2p^{b-k_2}).\end{aligned}$$

Proof. By 1.3 and Proposition 5.1, one has

$$\alpha_p(T,S)=\sum_{k\in\mathbb{Z}}p^{3k}\left(\frac{1}{2(1-(\frac{-1}{p})p^{-1})}K(p^{-k},p^{-k})+\frac{1}{2(1+(\frac{-1}{p})p^{-1})}K(p^{-k},up^{-k})\right)$$
$$+\frac{1}{4}\sum_{\substack{k_2<k_1\\ \beta_i=1\text{ or }u}}p^{2k_1+k_2}K(\beta_1p^{-k_1},\beta_2p^{-k_2}),$$

Where

$$K(t_1,t_2)=\int_{\mathrm{GL}_2(\mathbb{Z}_p)}J(t_1,t_2)\psi(-\operatorname{tr}T\,{}^tg\operatorname{diag}(t_1,t_2)g)dg$$

and

$$J(t_1,t_2)=\int_{(\mathbb{Z}_p^m)^2}\psi(\frac{1}{2}\operatorname{tr}{}^tg\begin{pmatrix}t_1&0\\0&t_2\end{pmatrix}g\begin{pmatrix}(x,x)&(x,y)\\(x,y)&(y,y)\end{pmatrix})\,dxdy.$$

A substitution of $\begin{pmatrix}x\\y\end{pmatrix}$ by $g\begin{pmatrix}x\\y\end{pmatrix}$ gives

$$\begin{aligned}J(t_1,t_2)&=\int_{(\mathbb{Z}_p^m)^2}\psi(\frac{1}{2}\operatorname{tr}\begin{pmatrix}t_1&0\\0&t_2\end{pmatrix}\begin{pmatrix}(x,x)&(x,y)\\(x,y)&(y,y)\end{pmatrix})dx\,dy\\&=\int_{(\mathbb{Z}_p^m)^2}\psi(t_1q(x))\psi(t_2q(y))\,dx\,dy\\&=J(t_1)J(t_2).\end{aligned}$$

In particular, $J(t_1,t_2)$ does not depend on g. Direct calculation gives

$$\begin{aligned}&\int_{\mathrm{GL}_2(\mathbb{Z}_p)}\psi(-\operatorname{tr}T\,{}^tg\operatorname{diag}(t_1,t_2)g)dg\\&=I^*(-\alpha_1p^at_1,-\alpha_1p^at_2,-\alpha_2p^bt_1,-\alpha_2p^bt_2).\end{aligned}$$

Finally, $J(\beta p^{-k})=(\frac{-1}{p})^{l(k,1)}J(p^{-k})$ by Corollary 2.3. Now putting pieces together, one proves the lemma.

Lemma 7.6. *Let the notation be as above.*

(1) *One has*

$$J(p^{-k})^2=p^{\sum_{l_i<k}(l_i-k)}(\frac{-1}{p})^{l(k,1)}.$$

(2) *One has*

$$(\frac{-1}{p})^{l(k,1)} R(k) = \begin{cases} (1-p^{-1})(\delta^+(k)+p^{-1}\delta^-(k)) & \text{if } k \le a, \\ p^{-1}(1-p^{-1})\delta^-(k) & \text{if } k = 1+a \& a < b, \\ -p^{-2}((\frac{-\alpha_1\alpha_2}{p})\delta^+(k)+\delta^-(k)) & \text{if } k = 1+a \& a = b, \\ 0 & \text{if } k > 1+a. \end{cases}$$

Proof. Claim (1) follows from Lemma 2.2 directly. When $k \le a$, one has by Corollary 6.3

$$I^*(-\alpha_1 p^{a-k}, -\beta\alpha_1 p^{a-k}, -\alpha_2 p^{b-k}, -\alpha_2 \beta p^{b-k}) = (1-p^{-1})^2(1+p^{-1}).$$

So

$$\begin{aligned} R(k) &= (1-p^{-1})^2(1+p^{-1})\{\frac{1}{2(1-(\frac{-1}{p})p^{-1})} + \frac{(-1)^{l(k,1)}}{2(1+(\frac{-1}{p})p^{-1})}\} \\ &= (1-p^{-1})(\delta^+(k)+(\frac{-1}{p})p^{-1}\delta^-(k)). \end{aligned}$$

Notice that

$$(\frac{-1}{p})^{l(k,1)}(x\delta^+(k)+y(\frac{-1}{p})\delta^-(k)) = x\delta^+(k)+y\delta^-(k) \tag{7.8}$$

for any numbers x and y. So

$$(\frac{-1}{p})^{l(k,1)} R(k) = (1-p^{-1})(\delta^+(k)+p^{-1}\delta^-(k)).$$

When $k = 1+a$ and $a < b$, one has $b-k \ge 0$. In this case, Corollary 6.4 implies

$$\begin{aligned} &I^*(-\alpha_1 p^{a-k}, -\beta\alpha_1 p^{a-k}, -\alpha_2 p^{b-k}, -\beta\alpha_2 p^{b-k}) \\ &= p^{-1}(1-p^{-1})(\frac{-\beta}{p})\{1-(\frac{-\beta}{p})p^{-1}\}. \end{aligned}$$

Now simple calculation (using (7.8)) gives the desired formula for this case. The case where $k = 1+a$ and $a = b$ follows similarly from Corollary 6.6 and Lemma 6.1

Finally, when $k > 1+a$, one has $a-k \le -2$. So Corollary 6.3 implies $R(k) = 0$.

Next come to the second sum in (7.7). First we renormalize the factors involved. Set for an integer k

$$\theta(k) = \begin{cases} 1 & \text{if } l(k,1) \text{ is even,} \\ \delta_p & \text{if } l(k,1) \text{ is odd.} \end{cases}$$

Here δ_p is defined by (1.9). The following trivial observation will be frequently used in the computations below.

$$\theta(k)^{-1}(x\delta^+(k) + y\delta_p\delta^-(k)) = x\delta^+(k) + y\delta^-(k), \tag{7.9}$$

where x and y are any complex numbers and k is any integer. Denote

$$R^*(k_1,k_2) = (\theta(k_1)\theta(k_2))^{-1}R(k_1,k_2) \tag{7.10}$$

and

$$J^*(k_1,k_2) = \theta(k_1)\theta(k_2)J(p^{-k_1})J(p^{-k_2}).$$

Then

$$J^*(k_1,k_2) = v_{k_1}v_{k_2}p^{\frac{1}{2}[\sum_{l_i<k_1}(l_i-k_1)+\sum_{l_i<k_2}(l_i-k_2)]} \tag{7.11}$$

where v_k is defined by (1.8), and

$$J^*(k_1,k_2)R^*(k_1,k_2) = J(p^{-k_1})J(p^{-k_2})R(k_1,k_2). \tag{7.12}$$

The following lemma follows directly from Corollary 6.3.

Lemma 7.7. (1) *If $k_2 < k_1 \le a$, then*

$$R^*(k_1,k_2) = (1-p^{-1})^2(1+p^{-1})\delta^+(k_1)\delta^+(k_2).$$

(2) *If $k_1 > 1+b$ or $k_2 > 1+a$, then $R^*(k_1,k_2) = 0$.*

Lemma 7.8. (1) *When $k_1 = 1+a > k_2$ and $a < b$, one has*

$$R^*(k_1,k_2) = (1-p^{-1})\delta^+(k_2)\left((\frac{\alpha_1}{p})\frac{1}{\sqrt{p}}\delta^-(k_1) - \frac{1}{p^2}\delta^+(k_1)\right).$$

(1′) *When $k_1 = 1+a > k_2$ and $a = b$, one has*

$$R^*(k_1,k_2) = p^{-1}(1-p^{-1})\left((\frac{-\alpha_1\alpha_2}{p}) - \frac{1}{p}\right)\delta^+(k_1)\delta^+(k_2).$$

(2) *When* $1+b > k_1 > 1+a > k_2$, *one has*

$$R^*(k_1, k_2) = (1-p^{-1})^2 p^{\frac{a-k_1}{2}} \delta^+(k_2) g(k)$$

(3) *When* $1+b > k_1 > 1+a = k_2$, *one has*

$$R^*(k_1, k_2) = (1-p^{-1}) p^{\frac{a-k_1}{2}} f_1(\alpha_1 p^a) g(k)$$

(4) *When* $k_1 = 1+b > 1+a > k_2$, *one has*

$$R^*(k_1, k_2) = (1-p^{-1}) p^{\frac{a-b-1}{2}} \delta^+(k_2) \cdot \begin{cases} f_1(\alpha_2 p^b) & \text{if } a-b \text{ is odd,} \\ (\frac{\alpha_1}{p}) f_2(\alpha_2 p^b) & \text{if } a-b \text{ is even.} \end{cases}$$

(5) *When* $k_1 = 1+b > 1+a = k_2$, *one has*

$$R^*(k_1, k_2) = (1-p^{-1}) p^{\frac{a-b-1}{2}} f_1(\alpha_1 p^a) \cdot \begin{cases} f_1(\alpha_2 p^b) & \text{if } a-b \text{ is odd,} \\ (\frac{\alpha_1}{p}) f_2(\alpha_2 p^b) & \text{if } a-b \text{ is even.} \end{cases}$$

Proof. The proof is case by case verification using results in section 6. To save space, we write

$$I(\beta_1, \beta_2) = I^*(-\alpha_1\beta_1 p^{a-k_1}, -\alpha_1\alpha_2 p^{a-k_2}, -\alpha_2\beta_1 p^{b-k_1}, -\alpha_2\beta_2 p^{b-k_2}).$$

When $k_1 = 1+a > k_2$ and $a < b$, one has $a - k_1 = -1$, $a - k_2 \geq 0$, $b - k_1 \geq 0$, and $b - k_2 \geq 0$. So Lemma 2.1 and Corollary 6.4 imply

$$I(\beta_1, \beta_2) = (1-p^{-1})(\delta_p(\frac{\alpha_1\beta_1}{p})\frac{1}{\sqrt{p}} - \frac{1}{p^2}).$$

Since $(\frac{u}{p}) = -1$, one has then

$$\begin{aligned} R(k_1, k_2) &= \frac{1}{4} \sum_{\beta_i = 1 \text{ or } u} (\frac{\beta_1}{p})^{l(k_1,1)} (\frac{\beta_2}{p})^{l(k_2,1)} I(\beta_1, \beta_2) \\ &= (1-p^{-1})\delta^+(k_2) \left((\frac{\alpha_1}{p}) \frac{1}{\sqrt{p}} \delta_p \delta^-(k_1) - \frac{1}{p^2} \delta^+(k_1) \right). \end{aligned}$$

Applying (7.9), one proves (1).

When $k_1 = 1 + a > k_2$ and $a = b$, one has $b - k_1 = a - k_1 = -1$, and $a - k_2 = b - k_2 \geq 0$. So Lemma 2.1 and Corollary 6.4 imply

$$\begin{aligned} I(\beta_1, \beta_2) &= (1 - p^{-1}) I^*(-\alpha_1\beta_1 p^{-1}) I(-\alpha_2\beta_1 p^{-1}) + p^{-1}(1 - p^{-1}) I^*(-\alpha_2\beta_1 p^{-1}) \\ &= p^{-1}(1 - p^{-1})((\frac{-\alpha_1\alpha_2}{p}) - p^{-1}) \end{aligned}$$

Here we have used $\delta^2 = (\frac{-1}{p})$. Plugging this into $R^*(k_1, k_2)$ and applying (7.9), one obtains (1′).

When $1 + b > k_1 > 1 + a > k_2$, one has $a - k_1 \leq -2$, $a - k_2 \geq 0$, $b - k_1 \geq 0$, and $b - k_2 \geq 0$. So Lemma 2.1 and Corollary 6.3 imply

$$\begin{aligned} I(\beta_1, \beta_2) &= (1 - p^{-1})^2 I(-\alpha_1\beta_1 p^{a-k_1}) \\ &= (1 - p^{-1})^2 p^{\frac{a-k_1}{2}} \begin{cases} 1 & \text{if } a - k_1 \text{ is even,} \\ \delta_p(\frac{\alpha_1\beta_1}{p}) & \text{if } a - k_1 \text{ is odd.} \end{cases} \end{aligned}$$

Plugging this into $R^*(k_1, k_2)$ and applying (7.9), one obtains (2).

When $1 + b > k_1 > 1 + a = k_2$, one has $a - k_1 \leq -2$, $a - k_2 = -1$, $b - k_1 \geq 0$, and $b - k_2 \geq 0$. So Lemma 2.1 and Corollary 6.3 imply

$$\begin{aligned} I(\beta_1, \beta_2) &= (1 - p^{-1}) I(-\alpha_1\beta_1 p^{a-k_1}) I^*(-\alpha_1\beta_2 p^{-1}) \\ &= (1 - p^{-1}) p^{\frac{a-k_1}{2}} (\delta_p(\frac{\alpha_1\beta_2}{p}) \frac{1}{\sqrt{p}} - \frac{1}{p}) \\ &\quad \cdot \begin{cases} 1 & \text{if } a - k_1 \text{ is even,} \\ \delta_p(\frac{\alpha_1\beta_1}{p}) & \text{if } a - k_1 \text{ is odd.} \end{cases} \end{aligned}$$

Plugging this into $R^*(k_1, k_2)$ and applying (7.9), one obtains (3).

When $1 + b = k_1 > 1 + a > k_2$, one has $a - k_1 \leq -2$, $b - k_1 = -1$, $a - k_2 \geq 0$, and $b - k_2 \geq 0$. So Lemma 2.1 and Corollary 6.3 imply

$$\begin{aligned} I(\beta_1, \beta_2) &= (1 - p^{-1}) I(-\alpha_1\beta_1 p^{a-k_1}) I^*(-\alpha_2\beta_1 p^{-1}) \\ &= (1 - p^{-1}) p^{\frac{a-b-1}{2}} (\delta_p(\frac{\alpha_2\beta_1}{p}) \frac{1}{\sqrt{p}} - \frac{1}{p}) \\ &\quad \cdot \begin{cases} 1 & \text{if } a - k_1 \text{ is even} \\ \delta_p(\frac{\alpha_1\beta_1}{p}) & \text{if } a - k_1 \text{ is odd} \end{cases} \\ &= (1 - p^{-1}) p^{\frac{a-b-1}{2}} \begin{cases} \delta_p(\frac{\alpha_2\beta_1}{p}) \frac{1}{\sqrt{p}} - \frac{1}{p} & \text{if } a + b \text{ is odd,} \\ (\frac{\alpha_1}{p})\{(\frac{-\alpha_2}{p}) \frac{1}{\sqrt{p}} - \delta_p(\frac{\beta_1}{p}) \frac{1}{p}\} & \text{if } a - b \text{ is even.} \end{cases} \end{aligned}$$

Here we have used $\delta_p^2 = (\frac{-1}{p})$. Plugging this into $R^*(k_1, k_2)$ and applying (7.9), one obtains (4).

Finally, when $1+b=k_1>1+a=k_2$, one has $a-k_1\le -2$, $b-k_1=-1$, $a-k_2=-1$, and $b-k_2\ge 0$. So Lemma 2.1 and Corollary 6.3 imply

$$\begin{aligned} I(\beta_1,\beta_2) &= I(-\alpha_1\beta_1 p^{a-k_1})I^*(-\alpha_2\beta_1 p^{-1})I^*(-\alpha_1\beta_2 p^{-1}) \\ &= p^{\frac{a-b-1}{2}}(\delta_p(\frac{\alpha_1\beta_2}{p})\frac{1}{\sqrt{p}}-\frac{1}{p}) \\ &\quad \cdot \begin{cases} \delta_p(\frac{\alpha_2\beta_1}{p})\frac{1}{\sqrt{p}}-\frac{1}{p} & \text{if } a+b \text{ is odd} \\ (\frac{\alpha_1}{p})\{(\frac{-\alpha_2}{p})\frac{1}{\sqrt{p}}-\delta_p(\frac{\beta_1}{p})\frac{1}{p}\} & \text{if } a-b \text{ is even.} \end{cases} \end{aligned}$$

Plugging this into $R^*(k_1,k_2)$ and applying (7.9), one obtains (5).

Proof of Theorem 7.1 Now the proof of Theorem 7.1 becomes tedious but easy calculation. We may assume $l=r=0$ by Lemma 7.4. We verify the case $a=b$ and omit the more complicated case $a\ne b$ for the benefit of the reader. By Lemma 7.6, one has

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} p^{3k}J(p^{-k})^2R(k) \\ &= (1-p^{-1})\sum_{k\le 0}p^{3k}+(1-p^{-1})\sum_{0<k\le a}p^{k+2d(k)}(\delta^+(k)+p^{-1}\delta^-(k)) \\ &\qquad - p^{-1+a+2d(1+a)}((\frac{-\alpha_1\alpha_2}{p})\delta^+(1+a)+\delta^-(1+a)) \\ &= (1-p^{-1})(1-p^{-3})^{-1}+(1-p^{-1})I_{2,8}+I_{2,6} \end{aligned}$$

By Lemma 7.2 and (7.11) – (7.12), one has

$$\begin{aligned} &\sum_{k_1>k_2} p^{2k_1+k_2}J(p^{-k_1})J(p^{-k_2})R(k_1,k_2) \\ &= \sum_{k_2<k_1\le 0}+\sum_{k_2\le 0<k_1\le a}+\sum_{k_2\le 0,\, k_1=a+1}+\sum_{0<k_2<k_1\le a}+\sum_{0<k_2\le a<k_1=a+1} \end{aligned}$$

where every sum is on $v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2)$. By Lemma 7.7, one has

$$\begin{aligned} &\sum_{k_2<k_1\le 0} v_{k_1}v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2) \\ &= (1-p^{-1})^2(1+p^{-1})\sum_{k_2<k_1\le 0}p^{2k_1+k_2} \\ &= p^{-1}(1-p^{-1})(1+p^{-1})(1-p^{-3})^{-1}. \end{aligned}$$

Notice that

$$p^{-1}(1-p^{-1})(1+p^{-1})(1-p^{-3})^{-1}+(1-p^{-1})(1-p^{-3})^{-1}=1.$$

By Lemmas 7.7, one has

$$\begin{aligned}&\sum_{k_2\le 0<k_1\le a} v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2)\\&=\sum_{0<k_1\le a} v_{k_1}p^{k_1+d(k_1)}\delta^+(k_1)(1-p^{-1})^2(1+p^{-1})\sum_{k_2\le 0}p^{k_2}\\&=I_{1,1}\end{aligned}$$

Similar calculation using Lemmas 7.8(1′) and 7.7 gives

$$\sum_{k_2\le 0,\,k_1=a+1} v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2)=I_{1,2}$$
$$\sum_{0<k_2<k_1\le a} v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2)=(1-p^{-1})I_{2,1}$$

and

$$\sum_{0<k_2\le a<k_1=a+1} v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2)=(1-p^{-1})I_{2,2}$$

Putting everything together and applying Lemma 7.5, one obtains

$$\alpha_p(T,S)=1+I_{1,1}+I_{1,2}+(1-p^{-1})(I_{2,1}+I_{2,2}+I_{2,6}+I_{2,8})+p^{-1}I_{2,6}.$$

This verifies (7.6) in the special case ($l=r=0$, $a=b$).

8. Examples.

In this section, we compute some interesting examples involving quaternion algebras. It turns out that the local densities related to split and ramified quaternion algebras are closely related to each other. Some results of this section will be used by Kudla and Rapoport to prove the main local identity in [Ku1] for $p|D(B)$ ([Ku1, Theorem 14.10]). Let B be a quaternion algebra over $\mathbb{Q}_p$. Fix a $\kappa\in\mathbb{Z}_p^*$, and let $V=\{x\in B:\ \operatorname{tr}x=0\}$ with the quadratic form $q(x)=-\kappa N(x)$, where N is the reduced norm. We will use upper script 's' or 'ra' indicates whether it is split or ramified. Let $\mathcal{O}_B$ be a maximal order of B. When $B=B^{ra}$, let $L^{ra}=V^{ra}\cap\mathcal{O}_B$. When $B=B^s$, we identify B with $M_2(\mathbb{Q}_p)$ in such a way that $\mathcal{O}_B=M_2(\mathbb{Z}_p)$. Let β be a fixed unit in $\mathbb{Z}_p$ such that $(\frac{\beta}{p})=-1$. We define

$$L_0^s=V^s\cap M_2(\mathbb{Z}_p)=\{\begin{pmatrix}a&b\\c&-a\end{pmatrix}\in M_2(\mathbb{Z}_p)\},$$

(8.1) $$L_1^s=\{\begin{pmatrix}a&b\\pc&-a\end{pmatrix}:\ a,b,c,\in\mathbb{Z}_p\},$$

and

$$L_2^s = \{\begin{pmatrix} pa & b \\ \beta b + pc & -pa \end{pmatrix} : \, a, b, c, \in \mathbb{Z}_p\}. \tag{8.2}$$

The lattice L_2^s consists of matrices in L_0^s whose reduction modulo p are in the image of $\mathbb{F}_{p^2} = \mathbb{F}_p(\beta)$ in $M_2(\mathbb{F}_p)$. Both L_1^s and L_2^s come naturally as the endomorphism rings of special Cartier $\mathcal{O}_{B^{ra}}$ modules over the algebraically closed field $\bar{\mathbb{F}}_p$ ([Ke, Section 2]). when $p \neq 2$, the corresponding matrices are

$$\begin{aligned} S_0^s &= \kappa \text{ diag}(1, 1, -1), \\ S_1^s &= \kappa \text{ diag}(1, p, -p), \\ S_2^s &= \kappa \text{ diag}(\beta, p^2, -\beta p^2), \\ S^{ra} &= \kappa \text{ diag}(\beta, p, -\beta p). \end{aligned} \tag{8.3}$$

When $p = 2$, the corresponding integral quadratic forms (see section 4) are

$$\begin{aligned} q_0^s &= \kappa(x_1^2 + y_1 y_2), \\ q_1^s &= \kappa(x_1^2 + 2y_1 y_2), \\ q_2^s &= \kappa(3x_1^2 - 3x_2^2 + 4x_3^2), \\ q^{ra} &= \kappa(3x_1^2 + 2(z_1^2 + z_1 z_2 + z_2^2). \end{aligned} \tag{8.4}$$

We are interested in computing the polynomial $\alpha(X, T^l, S^l)$ for $0 \leq l \leq 1$ and S being of the four matrices given by (8.3) and in their relations. We assume $n = \dim T \leq 2$ and $p \neq 2$ when $n = 2$. By Corollaries 3.2 and 7.2 and Theorem 4.1, one only needs to study the case $l = 0$.

Theorem 8.1**.** (1) *One has*

$$R_i(X, T, S_1^s) = p^i R_i(X, T, S_0^s),$$

and

$$R_i(X, T, S^{ra}) = (-p)^i R_i(X, T, S_0^s),$$

for $1 \leq i \leq 2$*.*

(2) *When* $a \geq 1$*, one has*

$$R_1(X, T, S_2^s) = p^{2i} R_1(X, T, S_0^s),$$

and

$$R_2(X, T, S_2^s) = p^4 R_2(X, T, S_0^s) + (p - p^3) X^2.$$

Proof. We assume $p \neq 2$ and leave the case $p = 2$ to the reader. A trivial but crucial observation is that $l(k, 1, S)$ is even if and only if k is even, independent of the choice among the four examples. Here we write $l(k, 1, S)$ for $l(k, 1)$ (see (1.6) for definition) to indicate its dependence on S, We will do the same for other notations. So the functions $\delta^{\pm}(k)$, $f_i(k)$, and $g(k)$ is the same for all the four examples. Direct calculation also gives

$$v(k, S) = \begin{cases} 1 & \text{if } k \text{ is even and } S = S_i^s, \\ -1 & \text{if } k \text{ is even and } S = S^{ra}, \\ (\frac{\kappa}{p}) & \text{if } k \text{ is odd and } S = S_i^s, \\ -(\frac{\kappa}{p}) & \text{if } k \text{ is odd and } S = S^{ra}, \end{cases} \tag{8.5}$$

with one exception: $v(1, S_2^s) = -(\frac{\kappa}{p})$. Finally, one has by calculation

$$d(k, S) = \begin{cases} -\frac{1}{2}k & \text{if } S = S_0^s, \\ -\frac{1}{2}k + 1 & \text{if } S = S_1^s \text{ or } S^{ra}, \\ \frac{1}{2} & \text{if } S = S_2^s \text{ and } k = 1, \\ -\frac{1}{2}k + 2 & \text{if } S = S_2^s \text{ and } k > 1. \end{cases} \tag{8.6}$$

Now the case T is a number is obvious by theorem 3.1. When $n = 2$, , one can now see easily by inspection that

$$I_{i,j}(X, T, S_1^s) = p^i I_{i,j}(X, T, S_0^s),$$

and

$$I_{i,j}(X, T, S^{ra}) = (-p)^i I_{i,j}(X, T, S_0^s),$$

for all i and j. This proves the first claim.

One has to be a little bit careful about $I_{i,j}(X, S_2^s)$ since $v(k, S_2^s)$ and $d(k, S_2^s)$ behave abnormally at $k = 1$. However, when $a \geq 1$, all the terms involving $k = 1$ is zero except in $I_{2,8}$. So the formulae $I_{ij}(X, T, S_2^s) = p^{2i} I_{ij}(X, T, S_0^s)$ are true for all (i, j) except $(2, 8)$. On the other hand,

$$\begin{aligned} I_{2,8}(X, S_2^s) &= p^{-1}p^2X^2 + \sum_{1<k\leq a} (\delta^+(k) + p^{-1}\delta^-(k))p^4X^{2k} \\ &= pX^2 - p^3X^2 + p^4 \sum_{0<k\leq a} (\delta^+(k) + p^{-1}\delta^-(k))X^{2k} \\ &= pX^2 - p^3X^2 + p^4 I_{2,8}(X, S_0^s). \end{aligned}$$

This proves the second claim.

Corollary 8.2. *For $t \in \mathbb{Z}_p$, one has*

$$\alpha(X, t, S^{ra}) + \alpha(X, t, S_1^s) = 2,$$

$$\alpha(X, t, S^{ra}) = -p\alpha(X, t, S_0^s) + 1 + p,$$

and

$$\alpha(X, t, S_i^s) = p^s \alpha(X, t, S_0^s) + 1 - p^i$$

for $0 \le i \le 2$.

Direct calculation using theorems 3.1 and 4.1 gives

Proposition 8.3. *Let $t = \alpha p^a$ with $\alpha \in \mathbb{Z}_p^*$ and $a \in \mathbb{Z}_{\ge 0}$.*

(1) *When a is odd, one has*

$$\alpha(X, t, S_0^s) = 1 + (1 - p^{-1}) \sum_{k=1}^{\frac{a-1}{2}} 2^{-k} X^{2k} - p^{-\frac{a+1}{2}-1} X^{a+1}.$$

(2) *When $p \neq 2$ and a is even, one has*

$$\alpha(X, t, S_0^s) = 1 + (1 - p^{-1}) \sum_{k=1}^{\frac{a}{2}} 2^{-k} X^{2k} + (\frac{\alpha\kappa}{p}) p^{-\frac{a}{2}-1} X^{a+1}.$$

(3) *When $p = 2$ and a is even, one has*

$$\alpha(X, t, S_0^s) = 1 + (1 - p^{-1}) \sum_{k=1}^{\frac{a-1}{2}} 2^{-k} X^{2k} + (\frac{-1}{\alpha\kappa}) 2^{-\frac{a}{2}-2} X^{a+2} + \delta_8(\alpha - \kappa) 2^{-\frac{a}{2}-2} X^{a+3},$$

where $(\frac{-1}{\alpha\kappa}) = \pm 1$ according to $\alpha\kappa \equiv \pm 1 \mod 4$.

$$\delta_8(x) = \begin{cases} 1 & \text{if } x \equiv 0 \mod 8, \\ -1 & \text{if } x \equiv 4 \mod 8, \\ 0 & \text{otherwise } . \end{cases}$$

From now on, we assume $p \neq 2$ and $n = 2$, i.e., T is a two-by-two nonsingular symmetric matrix over $\mathbb{Z}_p$. We may assume $T = \operatorname{diag}(\alpha_1 p^a, \alpha_2 p^b)$ with $a \le b$.

Corollary 8.4. *One has*

$$\begin{aligned}\alpha(X,T,S^{ra}) &= p^2\alpha(X,T,S_0^s)+1-p^2-(p+p^2)R_1(X,S_0^s),\\ \alpha(X,T,S_1^s) &= \alpha(X,T,S^{ra})+2pR_1(X,S_0^s),\end{aligned}$$

and

$$\alpha(X,T,S_2^s)=p^2\alpha(X,T,S^{ra})+(1+p)\left((1-p)(1+pX^2)+p^2R_1(X,S_0^s)\right).$$

So the four functions $\alpha(X,T,S)$ for $S=S_i^s$ or S^{ra} are all closely related, which is a somehow surprising outcome of this calculation. Thus it is sufficient to compute $R_i(X,S_0^s)$.

Theorem 8.5. *For $T=$ diag$(\alpha_1p^a,\alpha_2p^b)$, let $v_0=(\kappa\alpha_1/p)$, and*

$$v_1=\begin{cases}(\frac{\kappa\alpha_2}{p}) & \text{if } b\equiv 0 \mod 2,\\ (\frac{-\alpha_1\alpha_2}{p}) & \text{if } b\equiv 1 \mod 2.\end{cases}$$

(1) *If $a\equiv 0 \mod 2$, then*

$$R_1(X,S_0^s)=(1-p^{-2})\sum_{0<k\le\frac{a}{2}}p^kX^{2k}+p^{\frac{a}{2}}(1-p^{-1}v_0X)\sum_{a<k\le b}(v_0X)^k,$$

and

$$\begin{aligned}&R_2(X,S_0^s)\\ &=(1-p^{-1})p^{-2}\sum_{0<k\le\frac{a}{2}}p^kX^{2k}+(1-p^{-2}X^2)\sum_{0\le k<\frac{a}{2}}p^k(v_0X)^{a+b-2k}\\ &\quad+p^{\frac{a}{2}-1}X^2\sum_{a\le k<b}(v_0X)^k-p^{\frac{a}{2}-2}X^2\sum_{a\le k\le b}(v_0X)^k.\end{aligned}$$

(2) *If $a\equiv 1 \mod 2$, then*

$$R_1(X,S_0^s)=-1+(1-p^{-1}X^2)\sum_{0\le k\le\frac{a-1}{2}}p^kX^{2k}+v_1p^{\frac{a-1}{2}}X^{b+1},$$

and

$$\begin{aligned}R_2(X,S_0^s)&=(1-p^{-1})p^{-1}X^2\sum_{0\le k\le\frac{a-1}{2}}p^kX^{2k}-v_1p^{\frac{a-1}{2}}X^{b+1}\\ &\quad+(1-p^{-2}X^2)\sum_{0\le k\le\frac{a-1}{2}}v_1p^kX^{a+b-2k}.\end{aligned}$$

Proof. It follows from Theorem 7.1 by elementary but tedious calculation and is left to the reader. Another way to check it is to first compute $R_1(X, S_0^s)$ by definition, which is easy. Then applying the result and Kitaoka's formula as reformulated by Kudla (see Proposition 8.6 below) to Theorem 7.1, one can get a formula for $R_2(X, S_0^s)$.

The following two propositions follow from theorems 7.1 and 8.5 easily.

Proposition 8.6. *(Kitaoka, see [Ku1, Prop. 8.1])* (1) *When $a \equiv 0 \bmod 2$, one has*

$$\frac{\alpha(X, T, S_0^s)}{(1 - p^{-2}X^2)} = \sum_{0 \le k < \frac{a}{2}} p^k (X^{2k} + (v_0 X)^{a+b-2k}) + p^{\frac{a}{2}} \sum_{a \le k \le b} (v_0 X)^k.$$

(2) *When $a \equiv 1 \bmod 2$, one has*

$$\frac{\alpha(X, T, S_0^s)}{(1 - p^{-2}X^2)} = \sum_{0 \le k \le \frac{a-1}{2}} p^k (X^{2k} + v_1 X^{a+b-2k}).$$

Proposition 8.7. (1) *When $a \equiv 0 \bmod 2$, one has*

$$\begin{aligned} \alpha(X, T, S^{ra}) = &\sum_{0 \le k \le \frac{a}{2}} p^k (X^{2k} - (v_0 X)^{a+b+2-2k}) \\ &- \sum_{0 < k < \frac{a}{2}} p^{k+1} (X^{2k} - (v_0 X)^{a+b+2-2k}) \\ &+ (X^2 - 1) p^{\frac{a}{2}+1} \sum_{a \le k \le b} (v_0 X)^k. \end{aligned}$$

(2) *When $a \equiv 1 \bmod 2$, one has*

$$\begin{aligned} \alpha(X, T, S^{ra}) = &\sum_{0 \le k \le \frac{a+1}{2}} p^k (X^{2k} - v_1 X^{a+b+2-2k}) \\ &- \sum_{0 < k < \frac{a+1}{2}} p^{k+1} (X^{2k} - v_1 X^{a+b+2-2k}). \end{aligned}$$

For $T = \operatorname{diag}(\alpha_1 p^a, \alpha_2 p^b)$, we also denote $\alpha(X, \alpha_1 p^a, \alpha_2 p^b, S)$ for $\alpha(X, T, S)$. Then we have the following two induction formulas which follow from Propositions 8.6 and 8.7 directly.

Proposition 8.8. *(Induction formula I) Set $\nu = \nu_a = ((-1)^a \kappa\alpha_1/p)$.*

(1) *If $B = B^s$ is split, then*

$$\alpha(X, \alpha_1 p^a, \alpha_2 p^{b+1}, S_0^s) - vX\alpha(X, \alpha_1 p^a, \alpha_2 p^b, S_0^s)$$
$$= \begin{cases} (1-vX)(1-p^{-2}X^2)\left(\sum_{0\le k<\frac{a}{2}} p^k X^{2k}\right) + p^{\frac{a}{2}} X^a & \text{if } a \equiv 0 \mod 2, \\ (1-vX)(1-p^{-2}X^2)\sum_{0\le k<\frac{a}{2}} p^k X^{2k} & \text{if } a \equiv 1 \mod 2. \end{cases}$$

(2) *If $B = B^{ra}$ is ramified over $\mathbb{Q}_p$, then*

$$\alpha(X, \alpha_1 p^a, \alpha_2 p^{b+1}, S^{ra}) - vX\alpha(X, \alpha_1 p^a, \alpha_2 p^b, S^{ra})$$
$$= \begin{cases} (1-vX)\left(\sum_{0\le k\le\frac{a}{2}} p^k X^{2k} - \sum_{0<k<\frac{a}{2}} p^{k+1} X^{2k}\right) & \\ \quad +(X^2-1)p^{\frac{a}{2}+1}X^a & \text{if } a \equiv 0 \mod 2, \\ (1-vX)\left(\sum_{0\le k\le\frac{a+1}{2}} p^k X^{2k} - \sum_{0<k<\frac{a+1}{2}} p^{k+1} X^{2k}\right) & \text{if } a \equiv 1 \mod 2. \end{cases}$$

(3) *The difference $\alpha(X, \alpha_1 p^a, \alpha_2 p^{b+1}, S) - vX\alpha(X, \alpha_1 p^a, \alpha_2 p^b, S)$ is independent of b in all the four examples.*

Proposition 8.9. *(Induction formula II)*

(1) *When $B = B^s$ is split, one has*

$$\frac{\alpha(X, \alpha_1 p^a, \alpha_2 p^{b+2}, S_0^s) - X^2\alpha(X, \alpha_1 p^a, \alpha_2 p^b, S_0^s)}{(1-p^{-2}X^2)}$$
$$= \begin{cases} (1-X^2)\sum_{0\le k<\frac{a}{2}} p^k X^{2k} + p^{\frac{a}{2}} X^a (1+v_0 X) & \text{if } a \equiv 0 \mod 2, \\ (1-X^2)\sum_{0\le k<\frac{a}{2}} p^k X^{2k} & \text{if } a \equiv 1 \mod 2. \end{cases}$$

(2) *When $B = B^{ra}$ is ramified, one has*

$$\frac{\alpha(X, \alpha_1 p^a, \alpha_2 p^{b+2}, S^{ra}) - X^2\alpha(X, \alpha_1 p^a, \alpha_2 p^b, S^{ra})}{(1-X^2)}$$
$$= \begin{cases} \sum_0^{\frac{a}{2}} p^k X^{2k} - \sum_0^{\frac{a}{2}} p^{k+1} X^{2k} - p^{\frac{a}{2}+1} X^a (1+v_0 X) & \text{if } a \equiv 0 \mod 2, \\ \sum_0^{\frac{a+1}{2}} p^k X^{2k} - \sum_0^{\frac{a+1}{2}} p^{k+1} X^{2k} & \text{if } a \equiv 1 \mod 2. \end{cases}$$

For $T = \operatorname{diag}(\alpha_1 p^a, \alpha_2 p^b)$. we define following Kudla ([Ku1, section 8])

$$\mu_p(T) = \begin{cases} v_0^b & \text{if } a \equiv 0 \text{ mod } 2, \\ v_1 & \text{if } a \equiv 1 \text{ mod } 2, \end{cases} \tag{8.7}$$

where v_0 and v_1 are given in Theorem 8.5. Kudla has shown that $\mu_p(T) = -1$ if and only if $\alpha(1, T, S_0^s) = 0$, while $\mu_p(T) = 1$ if and only if $\alpha(1, T, S_0^{ra}) = 0$. It is natural and interesting to study the derivatives $\alpha'(1, T, S_0^s)$ when $\mu_p(T) = -1$, and $\alpha'(1, T, S^{ra})$ when $\mu_p(T) = 1$([Ka, section 8], and [Me, IV.3]). By Propositions 8.8 and 8.9, they also have interesting induction relations. The following peculiar relation was first observed by Kudla and Meyer ([Me, p. 57]). Since there is a minor mistake in the original statement, we restate it here with a simple proof. We don't know if there is a conceptional explanation.

Theorem 8.10**.** *(Kudla and Meyer) Let* $T =$ *diag*$(\alpha_1 p^a, \alpha_2 p^b)$*. When* $\mu_p(T) =$ 1*, one has*

$$\alpha'(1, T, S^{ra}) = -(p+1)(a+b+2) + \frac{2p^3}{p^2-1}\alpha(1, T, S_0^s).$$

Proof. Taking derivative at $X = 1$ on both sides of the formulas in Proposition 8.7, one has when $\mu_p(T) = 1$

$$\begin{aligned} &\alpha'(1, T, S^{ra}) + (p+1)(a+b+2) \\ &= \begin{cases} 4\sum_1^{\frac{a+1}{2}} p^k & \text{if } a \equiv 1 \bmod 2 \,\&\, v_1 = 1, \\ 4\sum_1^{\frac{a}{2}} p^k + 2p^{\frac{a}{2}+1} & \text{if } a \equiv b \equiv 0 \bmod 2 \,\&\, v_0 = -1, \\ 4\sum_1^{\frac{a}{2}} p^k + 2p^{\frac{a}{2}+1}(b-a+1) & \text{if } a \equiv 0 \bmod 2 \,\&\, v_0 = 1. \end{cases} \end{aligned}$$

Comparing this with [Ku1, Corollary 8.4] (see also Proposition 8.6), one obtains the theorem.

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