NONVANISHING OF HECKE L–FUNCTIONS AND THE BLOCH-KATO CONJECTURE

BYOUNG DU KIM, RIAD MASRI, AND TONG HAI YANG

Abstract. In this paper we study the central values of L–functions associated to a large class of algebraic Hecke characters of imaginary quadratic fields. When these central values are nonzero, the Bloch-Kato conjecture predicts an exact formula for the algebraic parts of the central values in terms of periods and arithmetic data, most notably the Selmer groups corresponding to the Hecke characters. We investigate the nonvanishing of these central values, and prove the p-part of the Bloch-Kato conjecture in these cases for primes p which split in K.

1. Introduction and statements of results

Let K be an imaginary quadratic field of discriminant $-D$ with $D > 3$ and $D \equiv 3 \bmod 4$. Let $\mathcal{O}_K$ be the ring of integers of K, and let

$$\epsilon_D : (\mathcal{O}_K/\sqrt{-D})^* \to \{\pm 1\}$$

be the quadratic character of K induced by the Dirichlet character $(-D/\cdot)$. A *canonical* Hecke character (in the sense of Rohrlich [R, R2]) is a Hecke character ψ_k of K of weight $k \in \mathbb{Z}^+$, infinity type $(2k-1, 0)$, and conductor $\sqrt{-D}\mathcal{O}_K$ which satisfies the condition

$$\psi_k(\alpha\mathcal{O}_K) = \epsilon_D(\alpha)\alpha^{2k-1} \quad \text{for} \quad (\alpha\mathcal{O}_K, \sqrt{-D}\mathcal{O}_K) = 1.$$

Let $\mathrm{CL}(K)$ be the ideal class group of K, let $h(-D)$ be the class number of K, and let $\xi : \mathrm{CL}(K) \to \mathbb{C}^\times$ be a class group character of K. Then there are exactly $h(-D)$ canonical Hecke characters, and they are given by

$$\Psi_k := \{\psi_k\xi : \ \xi \in \mathrm{CL}(K)^\wedge\}.$$

The L–function of ψ_k is defined by

$$L(\psi_k, s) = \sum_{0 \neq \mathfrak{a} \subset \mathcal{O}_K} \psi_k(\mathfrak{a}) N_{K/\mathbb{Q}}(\mathfrak{a})^{-s}, \quad \mathrm{Re}(s) > k + \frac{1}{2}$$

where the sum is over nonzero integral ideals $\mathfrak{a}$. It is known that $L(\psi_k, s)$ has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation under $s \mapsto 2k - s$ with central value $L(\psi_k, k)$.

The canonical Hecke characters were first studied by Gross [Gr], who constructed a "canonical" elliptic $\mathbb{Q}$-curve $A(D)$ associated to ψ_1. In particular, he showed that

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the extended Hecke character $\chi_H := \psi_1 \circ N_{H/K}$ of the Hilbert class field H of K corresponds to a unique (up to H-isogeny) $\mathbb{Q}$-curve $A(D)/H$ whose L–function satisfies

$$L(A(D)/H, s) = L(\chi_H, s)L(\overline{\chi_H}, s) = \prod_{\xi \in \mathrm{CL}(K)^\wedge} L(\psi_1\xi, s)L(\overline{\psi_1}\xi, s).$$

He made the remarkable conjecture that the rank of $A(D)(H)$ is 0 or $2h(-D)$ depending on whether D is congruent to 7 or 3 mod 8, respectively. Because the conjecture predicts an *exact* formula for the rank, the curves $A(D)/H$ form an important test case for the Birch and Swinnerton-Dyer conjecture.

Gross proved the rank 0 case of his conjecture for $D = p$ a prime using descent theory. Montgomery and Rohrlich [MR] (extending earlier work of Rohrlich [R, R2]) proved the rank 0 case for all discriminants by showing that $L(\psi_1\xi, 1) \neq 0$ for $D \equiv 7 \pmod 8$ and applying a theorem of Rubin [Ru1]. Miller and Yang [MiY] proved the rank $2h(-D)$ case by showing that $L'(\psi_1\xi, 1) \neq 0$ for $D \equiv 3 \pmod 8$ and applying a theorem of Kolyvagin and Logachev [KL]. The even discriminant cases were also positively settled in [MR] and [MiY].

When $k \geq 1$ the nonvanishing of the central values $L(\psi_k, k)$ and their quadratic twists has been studied using a wide-range of techniques (see e.g. [RV2, RVY, Y, LX, Ma, Ma2]). In this paper we will study this problem from a somewhat different perspective. We will establish an asymptotic formula for the first moment of the central values $L(\psi_k, k)$ which is of independent interest. Our proof of the asymptotic formula relies on an "equidistribution theorem" for Heegner points. Such theorems exist in the literature. Our calculation is complicated by the fact that the test functions involved are not of rapid decay (see Theorem 5.1). We will then combine the asymptotic formula with subconvexity bounds to prove a nonvanishing theorem for certain subfamilies of canonical Hecke characters.

Given these nonvanishing theorems, it is natural to ask if they shed light on the arithmetic of the Selmer groups corresponding to the canonical Hecke characters. Recall that the Bloch-Kato conjecture [BK] predicts an exact formula for the algebraic part of $L(\psi_k, k)$ in terms of periods and arithmetic data, most notably the order of the Selmer group corresponding to ψ_k. We will prove the p-part of the Bloch-Kato conjecture in these cases for primes p which split in K, and combine this with nonvanishing theorems to prove a finiteness theorem for the corresponding Selmer groups.

In order to state our results we fix the following notation. Let $d \equiv 1 \mod 4$ be a squarefree integer coprime to D. The quadratic twist of ψ_k is defined by $\psi_{d,k} := (d/N_{K/\mathbb{Q}}(\cdot))\psi_k$. Clearly, there are exactly $h(-D)$ such characters and they are given by

$$\Psi_{d,k} := \{\psi_{d,k}\xi : \ \xi \in \mathrm{CL}(K)^\wedge\}.$$

For an integer $n \geq 1$, let $\mathrm{CL}_n(K)$ be the kernel of the n-th power map on $\mathrm{CL}(K)$, and let

$$\mathrm{CL}^{(n)}(K) := \mathrm{CL}(K)/\mathrm{CL}_n(K) \cong \mathrm{CL}(K)^n.$$

Define the natural subfamily for a fixed $\psi_{d,k} \in \Psi_{d,k}$

$$\Psi_{d,k}^{(2)} := \{\psi_{d,k}\xi : \ \xi \in \mathrm{CL}^{(2)}(K)^\wedge\} \subset \Psi_{d,k}$$

of canonical Hecke characters which differ by ideal class group characters which are trivial on the 2-torsion subgroup of the class group.

Define the theta function

$$\theta_{d,k}(z) := (2y)^{-k/2} \sum_{(n,d)=1} \left(\frac{d}{n}\right) H_k(n\sqrt{2y})e(n^2 z), \quad y = \mathrm{Im}(z) > 0$$

where H_k is a suitably normalized Hermite polynomial of degree k (see Remark 3.3) and $e(z) := e^{2\pi i z}$. The function $\theta_{d,k}$ is a weight $k + \frac{1}{2}$ modular form for $\Gamma_0(4d^2)$ with character $(d/\cdot)$.

Define the Peterson inner product

$$\langle \theta_{d,k}, \theta_{d,k} \rangle_{\mathrm{Pet}} := \int_{Y_0(4d^2)} |\theta_{d,k}(z)|^2 \, \mathrm{Im}(z)^{k+\frac{1}{2}} d\mu(z)$$

where the Poincaré measure $d\mu(z)$ is normalized so that the the open modular curve $Y_0(4d^2)$ has volume 1.

We will establish the following asymptotic formula for the first moment of central values associated to canonical Hecke characters. An outline of the proof is given in section 2.

Theorem 1.1. *Let $D \equiv 7 \mod 8$ be a positive, squarefree integer, and let $d \equiv 1 \mod 4$ be a squarefree integer coprime to D such that every prime divisor of $2d$ splits in K. Let $k \geq 1$ be an integer such that $\mathrm{sign}(d) = (-1)^{k-1}$. Then for all $\delta < 1/8$ we have*

$$\frac{1}{\#\mathrm{CL}^{(2)}(K)} \sum_{\psi_{d,k} \in \Psi_{d,k}^{(2)}} L(\psi_{d,k}, k) = c(k) L_D(1) \langle \theta_{d,k-1}, \theta_{d,k-1} \rangle_{\mathrm{Pet}} + O_{d,k,\delta}(D^{-\delta}) \qquad (1.1)$$

as $D \to \infty$. Here $c(k) := 2(8\pi)^{k-1}/(k-1)!$ and $L_D(s)$ is the L–function of the Dirichlet character $(-D/\cdot)$. Moreover,

$$\frac{1}{h(-D)} \sum_{\psi_{d,k} \in \Psi_{d,k}} L(\psi_{d,k}, k) = c(k) L_D(1) \langle \theta_{d,k-1}, \theta_{d,k-1} \rangle_{\mathrm{Pet}} + O_{d,k,\delta}(D^{-\delta}) \qquad (1.2)$$

as $D \to \infty$. The implied constants in the error terms $O_{d,k,\delta}$ are ineffective.

Remark 1.2. In [Ma], the second author proved a variant of the asymptotic formula in Theorem 1.1 (1.2) for $d = 1$ and k odd. The restriction to $d = 1$ was necessary to use a formula of Rodriguez-Villegas and Zagier [RV, RVZ] for the central value $L(\psi_k, k)$, and the *crucial* restriction to k odd was necessary to insure that the theta function appearing in this formula was cuspidal. See the discussion in section 2.

Remark 1.3. In [T, Theorems 1 and 2], Templier uses a different method to obtain asymptotic formulae for the first moment of central values of canonical Hecke L–functions analogous to those in Theorem 1.1. Whereas we use period relations, equidistribution, and a spectral regularization in the proof of Theorem 1.1 (see section 2), Templier uses the approximate functional equation and a subconvexity bound due to Burgess.

We will combine Theorem 1.1 with subconvexity bounds of Duke, Friedlander, and Iwaniec [DFI] to prove the following nonvanishing theorem for the subfamily $\Psi_{d,k}^{(2)}$.

Theorem 1.4. *Let assumptions be as in Theorem 1.1. Then for all $\delta < 1/60$,*

$$\#\{\psi_{d,k} \in \Psi_{d,k}^{(2)} : \ L(\psi_{d,k}, k) \neq 0\} \gg_{d,k,\delta} D^{\delta}$$

as $D \to \infty$. The implied constant in $\gg_{d,k,\delta}$ is ineffective.

We now turn to a discussion of our arithmetic results. Let ψ be an algebraic Hecke character of K of conductor f and infinity type $(2k-1, 0)$. Let $K(f)$ be the ray class field of K of conductor f, and let $G(f)$ be the Galois group of $K(f)/K$. Let p be a prime number not dividing f. By fixing an embedding $i_p : \bar{\mathbb{Q}} \to \mathbb{C}_p$ one can associate a p-adic Galois character of $\mathrm{Gal}(fp^\infty)$ to ψ. Let $\mathrm{Sel}_p(A_\psi/K)$ be the Selmer group associated to the Galois representation A_ψ defined by (8.1).

In the following theorem we will equate the p-adic valuations of the two quantities appearing in the Bloch-Kato conjecture for primes p which split in K.

Theorem 1.5. *Assume that $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K and is prime to $[K(f) : K]$. Then $L(\bar{\psi}, k) \neq 0$ if and only if $\mathrm{Sel}_p(A_\psi/K)$ is finite. If these two equivalent conditions are satisfied, then*

$$v_p(\# \mathrm{Sel}_p(A_\psi/K) \prod_{v|f} a_v) = v_p\left((\frac{\sqrt{-D}}{2\pi})^{-k+1} (1 - \frac{\psi(\mathfrak{p})}{p^k}) \frac{L_{f\bar{\mathfrak{p}}}(\bar{\psi}, k)}{\Omega^{2k-1}} \right)$$

where a_v is the Tamagawa number of the prime v, $L_{f\bar{\mathfrak{p}}}$ is the L-function with the Euler factors for the primes dividing $f\bar{\mathfrak{p}}$ removed, and Ω is the complex period of a CM elliptic curve of conductor f.

Remark 1.6. Results similar to Theorem 1.5 were proved by Guo [Gu1] and Han [Han] for imaginary quadratic fields of class number 1.

As almost an immediate consequence of the results stated above we will obtain the following finiteness theorem for the Selmer groups $\mathrm{Sel}_p(\psi_{d,k}/K)$.

Theorem 1.7. *Let $D \equiv 7 \mod 8$ be a positive, squarefree integer, and let $d \equiv 1 \mod 4$ be a squarefree integer coprime to D. Let $f = d\sqrt{-D}\mathcal{O}_K$, and let $p \nmid f$ be a prime number that splits in K and is prime to $[K(f) : K]$. Let $k \geq 1$ be an integer such that $\mathrm{sign}(d) = (-1)^{k-1}$.*

1. *If $(2k-1, h(-D)) = 1$ then for all $|d| \ll_{k,\epsilon} D^{\frac{1}{12}-\epsilon}$,*
$$\#\{\psi_{d,k} \in \Psi_{d,k} : \ \#\mathrm{Sel}_p(\psi_{d,k}/K) < \infty\} = h(-D).$$
2. *If every prime divisor of $2d$ splits in K then for all $\delta < 1/60$,*
$$\#\{\psi_{d,k} \in \Psi_{d,k}^{(2)} : \ \#\mathrm{Sel}_p(\psi_{d,k}/K) < \infty\} \gg_{d,k,\delta} D^{\delta}$$
as $D \to \infty$.

The implied constants in $\ll_{k,\epsilon}$ and $\gg_{d,k,\delta}$ are ineffective.

Finally, in the following corollary we will show that Theorem 1.7 can be viewed as a "higher weight" generalization of the rank 0 case of Gross's conjecture.

Corollary 1.8. *Let D and p be as in Theorem 1.7. Let $\mathfrak{p}$ be the prime of K above p fixed by the embedding i_p, and let $\mathrm{Sel}_{\mathfrak{p}}(A(D)/H)$ be the associated $\mathfrak{p}$-Selmer group of $A(D)/H$. Let $Ш(A(D)/H)$ be the Shafarevich-Tate group of $A(D)/H$. Then $\mathrm{Sel}_{\mathfrak{p}}(A(D)/H)$, $A(D)(H)$, and $Ш(A(D)/H)_{\mathfrak{p}}$ are finite.*

Organization. The paper is organized as follows. In section 2 we outline the proof of Theorem 1.1. In sections 3-7 we prove Theorems 1.1 and 1.4. In section 8 we prove Theorem 1.5. In section 9 we study the complex period appearing in Theorem 1.5. Finally, in section 10 we prove Theorem 1.7 and Corollary 1.8.

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2. Outline of the proof of Theorem 1.1

In recent years the problem of obtaining asymptotics for moments of L–functions has been studied using period relations of Waldspurger type to express the average of the central values of a family of L–functions as an average of a fixed automorphic function over special points on some variety. In many situations, an asymptotic formula can then be obtained using the distribution properties of the special points. See for example the work of Vatsal [V] on Mazur's conjecture, and the work of Michel and Venkatesh [MV, MV2] on nonvanishing of Rankin-Selberg L–functions. We will take a similar approach to prove Theorem 1.1.

In Theorem 3.2 and Lemma 3.4 we will establish a formula for the central value $L(\psi_{d,k}, k)$ of the form

$$L(\psi_{d,k}, k) = \kappa \#\mathrm{CL}_2(K) \left| \sum_{C \in \mathrm{CL}(K)^2} \frac{\theta_{d,k-1}(\tau_C)}{\psi_{d,k}(\bar{\mathfrak{a}})} \right|^2 \tag{2.1}$$

where κ is an explicit constant depending on d, D and k, $[\mathfrak{a}] = C$, and $\{\tau_C : C \in \mathrm{CL}(K)^2\}$ is a $\mathrm{CL}(K)^2$-orbit of Heegner points of discriminant $-D$ on $X_0(4d^2)$. This constitutes a slight generalization of the central value formulas in [Y, RVY].

In Theorem 3.5 we will use (2.1) to establish exact formula for the average of the central values of the form

$$\frac{1}{\#\mathrm{CL}^{(2)}(K)} \sum_{\psi_{d,k} \in \Psi_{d,k}^{(2)}} L(\psi_{d,k}, k) = c(k) L_D(1) \frac{1}{\#(\mathrm{CL}(K)^2)} \sum_{C \in \mathrm{CL}(K)^2} F_{d,k}(\tau_C) \tag{2.2}$$

where $F_{d,k} : \mathbb{H} \to \mathbb{R}_{\geq 0}$ is the C^∞, $\Gamma_0(4d^2)$-invariant defined by

$$F_{d,k}(z) := \mathrm{Im}(z)^{k-\frac{1}{2}} \left|\theta_{d,k-1}(z)\right|^2 .$$

We want to combine (2.2) with a (deep) equidistribution theorem of Harcos and Michel [HM] for Galois suborbits of Heegner points on modular curves to obtain the asymptotic formula in Theorem 1.1. However, the "test function" $F_{d,k}$ is in general *not* of rapid decay in the cusps of $X_0(4d^2)$ (see section 4, and in particular Proposition 4.4). We overcome this difficulty by using a careful spectral regularization to extend the Harcos-Michel theorem to include test functions satisfying a moderate growth condition in the cusps (see Theorem 5.1).

3. An exact formula for the average L–value

In [RVY, Theorem 3.2], Rodriguez-Villegas and Yang gave an explicit formula for the central value $L(\psi_{d,1}^{2k-1}, k)$. If $(2k-1, h(-D)) = 1$, the two families $\Psi_{d,1}^{2k-1}$ and $\Psi_{d,k}$ coincide. Otherwise, they are different. Here we modify the proof of [RVY, Theorem 3.2] to give a formula for $L(\psi_{d,k}, k)$ in general.

Let $\chi = \prod \chi_p$ be the 'canonical' additive character of $\mathbb{Q}\backslash\mathbb{Q}_{\mathbb{A}}$ such that $\chi_\infty(x) = e(x) = e^{2\pi i x}$. Here $F_{\mathbb{A}}$ is the adeles of a number field F and $F^*_{\mathbb{A}}$ is the ideles of F. Let $\chi_K = \chi \circ \mathrm{tr}_{K/\mathbb{Q}}$, and let $\epsilon(\frac{1}{2}, \psi_{d,k,w}, \frac{1}{2}\chi_{K_w})$ be the local root number of the local character $\psi_{d,k,w}$ at a prime w of K (with respect to the local additive character $\frac{1}{2}\chi_{K_w}$). Here we identify a Hecke character $\psi_{d,k}$ with its associated idele class character $\psi_{d,k} = \prod_w \psi_{d,k,w}$.

Lemma 3.1. *Let assumptions be as in Theorem 1.1.*

(1) *There are exactly $2^{-t(D)}h(-D)$ Hecke characters $\psi_{d,k} \in \Psi_{d,k}$ of K such that*

$$\prod_{w|p} \epsilon(\frac{1}{2}, \psi^{un}_{d,k,w}, \frac{1}{2}\chi_{K_w})\psi^{un}_{d,k,w}(\sqrt{-D}) = 1 = \epsilon_p(\frac{4}{D}), \tag{3.1}$$

for every prime $p \leq \infty$. Here $t(D)+1$ is the number of prime factors of D and $\psi^{un} = \psi/|\psi|$ is the unitarization of ψ. We denote Hecke characters satisfying (3.1) by $\psi_{D,d,k}$, which differ from each other by ideal class characters trivial on $\mathrm{CL}_2(K)$.

(2) *For each ideal class character ξ of K, we can choose a positive squarefree integer $\beta = \beta(\xi)$ such that*

$$\epsilon_p(\beta) = \begin{cases} \xi_w(\sqrt{-D}) & \text{if } p|D, \\ 1 & \text{if } p \nmid D. \end{cases} \tag{3.2}$$

For this β, we have

$$\prod_{w|p} \epsilon(\frac{1}{2}, \psi^{un}_{D,d,k,w}\xi_w, \frac{1}{2}\chi_{K_w})\psi^{un}_{D,d,k,w}\xi_w(\sqrt{-D}) = \epsilon_p(\frac{4}{D}\beta)$$

for every prime p.

(3) *For a Hecke character $\psi_{d,k} \in \Psi_{d,k}$, there is a positive squarefree integer $\beta = \beta(\psi_{d,k})$ such that*

$$\prod_{w|p} \epsilon(\frac{1}{2}, \psi^{un}_{d,k,w}, \frac{1}{2}\chi_{K_w})\psi^{un}_{d,k,w}(\sqrt{-D}) = \epsilon_p(\frac{4}{D}\beta).$$

Proof. (sketch) The proof is similar to [RVY, Lemma 3.1]. We give a sketch here for the convenience of the reader. Let ψ_0 be a fixed element in $\Psi_{d,k}$. Then every element in $\Psi_{d,k}$ has the form $\psi_0\xi$ with $\xi \in \mathrm{CL}(K)^\wedge$. Now [RVY, Lemma 2.1] implies that for a prime $p \le \infty$

$$\prod_{w|p} \epsilon(\frac{1}{2}, \psi^{un}_{0,w}\xi_w, \frac{1}{2}\chi_{K_w})(\psi^{un}_{0,w}\xi_w)(\sqrt{-D}) = \begin{cases} 1 & \text{if } p \nmid D, \\ \epsilon(\frac{1}{2}, \epsilon_p, \chi_p)(\psi^{un}_{0,\mathfrak{p}}\xi_{\mathfrak{p}})(\sqrt{-D}) & \text{if } p|D. \end{cases}$$

Here $\mathfrak{p} = w$ is the prime of K above p. Set for each $p|D$

$$\kappa_p = \epsilon(\frac{1}{2}, \epsilon_p, \chi_p)\psi^{un}_{0,\mathfrak{p}}(\sqrt{-D}) = \pm 1,$$

and

$$S = \{(c_p)_{p|D} : \ c_p = \pm 1, \quad \prod c_p = 1\}.$$

Notice that the same proof as that of [RVY, Lemma 2.1] gives the global root number

$$\epsilon(\frac{1}{2}, \psi^{un}_0\xi, \frac{1}{2}\chi_K) = (-1)^{k-1}\mathrm{sign}(d) = 1.$$

So $\prod \kappa_p = 1$ and $(\kappa_p) \in S$. The genus theory gives a bijection

$$F : (\mathrm{CL}(K)/\mathrm{CL}_2(K))^\wedge \to S, \quad \xi \mapsto (\xi_{\mathfrak{p}}(\sqrt{-D}))_{p|D} = (\xi(\mathfrak{p}))_{p|D}.$$

So there are exactly $2^{-t(D)}h(-D)$ ideal class characters ξ such that $F(\xi) = (\kappa_p)$, and these characters ξ differ from each other by ideal class characters which are trivial on $\mathrm{CL}_2(K)$. For such an ideal class character ξ, $\psi_{d,k} = \psi_0\xi$ satisfies (3.1). This proves (1). Claims (2) and (3) follow from (1) directly.

□

Notice that every prime factor of β is split in K by Lemma 3.1, and $\beta \in \mathbb{Q}^*/N_{K/\mathbb{Q}}K^*$ is uniquely determined by $\psi_{d,k} = \psi_{D,d,k}\xi$ (since the class of β is uniquely determined by (3.2)) and depends only on the family $\{\psi_{d,k}\xi : \xi \in \mathrm{CL}^{(2)}(K)^\wedge\}$. We let $\Psi^{(2)}_{d,k,\beta}$ denote this subfamily of Hecke characters. We mention that [RVY, Lemma 3.1] is not correct as stated: not every β above can be represented by a factor D_1 of D in its class.

By applying [RVY, Theorem 2.5] to $(\psi^{un}_{D,d,k}\xi, 1, \delta = \sqrt{-D}, \alpha = \frac{4}{D}\beta, \psi)$ (see also [MaY, Theorem 3.7]), we obtain the following theorem.

Theorem 3.2. *Let $\psi_{d,k} \in \Psi_{d,k}$ and let $\beta = \beta(\psi_{d,k})$ be associated to $\psi_{d,k}$ as in Lemma 3.1. Then the central value*

$$L(\psi_{d,k}, k) = \kappa \#\mathrm{CL}_2(K) \left| \sum_{C \in \mathrm{CL}^{(2)}(K)} \frac{\theta_{d,k-1}(\tau_{\beta,\mathfrak{a}^2})}{\psi_{d,k}(\bar{\mathfrak{a}})} \right|^2 .$$

Here

$$\kappa := \frac{\pi^k}{\sqrt{2D}(k-1)!}\left(\frac{\sqrt{D}}{\beta d^2}\right)^{k-\frac{1}{2}}$$

and $\tau_{\beta,\mathfrak{a}^2}$ is a Heegner point on $X_0(4d^2)$ of discriminant $-D$ given as follows. Fix a square root r of $-D \mod 16\beta d^2$, and for a primitive integral ideal $\mathfrak{a} \in C^{-1}$, write

$$\mathfrak{a}^2 = [a^2, \frac{b+\sqrt{-D}}{2}], \quad a, b \in \mathbb{Z}$$

with $a = N\mathfrak{a}$, and b satisfying

$$b^2 \equiv -D \mod 16\beta d^2 a^2, \quad b \equiv r \mod 8\beta d^2.$$

Then

$$\tau_{\beta,\mathfrak{a}^2} = \frac{b+\sqrt{-D}}{8\beta d^2 a^2}.$$

Remark 3.3. By [RVY], the polynomial $H_k(x)$ occuring in the definition of $\theta_{d,k}$ is given by the equation

$$\frac{1}{2^k}\left(x - \frac{1}{2\pi}\frac{d}{dx}\right)^k e^{-\pi x^2} = H_k(x) e^{-\pi x^2}.$$

We now show that $H_k(x)$ can be expressed as

$$H_k(z) = \frac{1}{(\sqrt{8\pi})^k} \sum_{0 \le j \le k/2} \frac{k!}{j!(k-2j)!}(-1)^j (\sqrt{8\pi} z)^{k-2j}.$$

First, $H_k(x)$ is determined by the following recurrence formula

$$H_k(x) = xH_{k-1}(x) - \frac{k-1}{4\pi} H_{k-2}(x), \quad H_0(x) = 1, \quad H_1(x) = x. \tag{3.3}$$

On the other hand, the classical k-th Hermite polynomial

$$\widetilde{H}_k(x) = \sum_{0 \le j \le k/2} \frac{k!}{j!(k-2j)!}(-1)^j (2x)^{k-2j} \tag{3.4}$$

is determined by the recurrence formula

$$\widetilde{H}_k(x) = 2x\widetilde{H}_{k-1}(x) - 2(k-1)\widetilde{H}_{k-2}(x), \quad \widetilde{H}_0(x) = 1, \quad \widetilde{H}_1(x) = 2x.$$

It is now easy to check that

$$\frac{\widetilde{H}_k(\sqrt{2\pi}x)}{(\sqrt{8\pi})^k}$$

satisfies (3.3), and so

$$H_k(x) = \frac{\widetilde{H}_k(\sqrt{2\pi}x)}{(\sqrt{8\pi})^k} \tag{3.5}$$

as claimed.

In the next lemma we show that the Heegner points appearing in Theorem 3.2 form a $\mathrm{CL}(K)^2$-suborbit of Heegner points on $X_0(4d^2)$. This will be crucial for our application of Theorem 5.1.

Lemma 3.4. *Let $r^2 \equiv -D \mod 16\beta d^2$. Then*

(1) $\mathcal{H}_{\beta,D}^{(r)} = \{\tau_{\beta,\mathfrak{a}^2} : [\mathfrak{a}] \in \mathrm{CL}^{(2)}(K)\}$ *is a $\mathrm{CL}(K)^2$-orbit of Heegner points on $X_0(4d^2)$.*

(2) $\mathcal{H}_D^{(r)} = \bigcup_{\beta=\beta(\psi_{d,k})} \mathcal{H}_{\beta,D}^{(r)}$ *is a $\mathrm{CL}(K)$-orbit of Heegner points on $X_0(4d^2)$.*

(3) *When the square roots r of $-D \mod 16d^2$ change modulo $8d^2$, $\mathcal{H}_D^{(r)}$ gives all Heegner points on $X_0(4d^2)$ of discriminant $-D$.*

Proof. Let N be a positive integer. Heegner points on $X_0(N)$ of discriminant $-D$ exist if and only if every prime factor of N is split or ramified in $K = \mathbb{Q}(\sqrt{-D})$, and they are given as follows. Fix a square root $r \mod 2N$ of $-D \mod 4N$. For a primitive ideal $\mathfrak{a}$, write

$$\mathfrak{a} = [a, \frac{b+\sqrt{-D}}{2}], \quad a = \mathfrak{a}\bar{\mathfrak{a}} > 0,\ b \in \mathbb{Z}$$

with

$$b \equiv r \mod 2N, \quad b^2 \equiv -D \mod 4Na.$$

Notice that the congruence $b^2 \equiv -D \mod 4Na$ is automatically satisfied when $(4N, a) = 1$. Then

$$\tau_{\mathfrak{a}} = \tau_{\mathfrak{a}}^{(r)} = \frac{b+\sqrt{-D}}{2Na}$$

is a Heegner point on $X_0(N)$. In terms of quadratic forms, this Heegner point is associated to $Nax^2 - bxy + cy^2$. It is known that $\tau_{\mathfrak{a}}$ depends only on the ideal class of $\mathfrak{a}$ and $r \mod 2N$. So we can denote it by $\tau_{[\mathfrak{a}]}$ or $\tau_{[\mathfrak{a}]}^{(r)}$, depending on whether r is important in the context. The ideal class group $\mathrm{CL}(K)$ acts on Heegner points via

$$[\mathfrak{b}].\tau_{[\mathfrak{a}]}^{(r)} = \tau_{[\mathfrak{a}\mathfrak{b}]}^{(r)}.$$

In our case, every prime factor of $4\beta d^2$ is split or ramified in K, and so $\tau_{\beta,\mathfrak{a}^2} = \tau_{[\mathfrak{a}^2]}^{(r)}$ is a Heegner point on $X_0(4\beta d^2)$. Projecting to $X_0(4d^2)$, it becomes a Heegner point $\tau_{\beta,\mathfrak{a}^2} = \tau_{[\mathfrak{b}\mathfrak{a}^2]}^{(r)}$ depending only on $r \mod 8d^2$. Here $\beta\mathcal{O}_K = \mathfrak{b}\bar{\mathfrak{b}}$ for some integral ideal $\mathfrak{b}$. So $\mathcal{H}_{\beta,D}^{(r)}$ is a $\mathrm{CL}(K)^2$-orbit of Heegner points on $X_0(4d^2)$. This proves (1).

Suppose $\mathcal{H}_{\beta_1,D}^{(r)} \cap \mathcal{H}_{\beta_2,D}^{(r)}$ is not empty. Then there are ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ such that $\tau_{\beta_1,\mathfrak{a}_1^2} = \tau_{\beta_2,\mathfrak{a}_2^2}$ in $X_0(4d^2)$, i.e.,

$$\tau_{[\mathfrak{b}_1\mathfrak{a}_1^2]}^{(r)} = \tau_{[\mathfrak{b}_2\mathfrak{a}_2^2]}^{(r)}.$$

Then $[\mathfrak{b}_1\mathfrak{a}_1^2] = [\mathfrak{b}_2\mathfrak{a}_2^2]$, i.e., $\mathfrak{b}_1\mathfrak{a}_1^2 = z\mathfrak{b}_2\mathfrak{a}_2^2$ for some $z \in K^*$. Taking the norm, we have

$$\beta_1 a_1^2 = N(z)\beta_2 a_2^2$$

and thus $\epsilon_p(\beta_1) = \epsilon_p(\beta_2)$ for every prime $p \leq \infty$. This implies by Lemma 3.1 that β_1 and β_2 are associated to the same class of $\psi_{d,k}$. Here two Hecke character $\psi_1, \psi_2 \in \Psi_{d,k}$ are in the same class if they differ by a character of $\mathrm{CL}^{(2)}(K)$. Now a simple counting gives (2). Claim (3) follows directly from (2). $\square$

In the following theorem we give an exact formula for the average of the central values $L(\psi_{d,k}, k)$.

Theorem 3.5. *Let assumptions be as in Theorem 1.1. Let $\Psi^{(2)}_{d,k,\beta}$ be a subfamily of $\Psi_{d,k}$ as defined above. Then*

$$\frac{1}{\#\mathrm{CL}^{(2)}(K)} \sum_{\psi_{d,k}\in\Psi^{(2)}_{d,k,\beta}} L(\psi_{d,k}, k) = c(k)L_D(1)\frac{1}{\#\mathrm{CL}(K)^2} \sum_{C\in\mathrm{CL}(K)^2} F_{d,k}(\tau_{\beta,C}) \tag{3.6}$$

where $c(k) := 2(8\pi)^{k-1}/(k-1)!$ and $\{\tau_{\beta,C} : C \in \mathrm{CL}(K)^2\}$ is a $\mathrm{CL}(K)^2$-orbit of Heegner points on $X_0(4d^2)$ associated to β. Moreover,

$$\frac{1}{h(-D)} \sum_{\psi_{d,k}\in\Psi_{d,k}} L(\psi_{d,k}, k) = c(k)L_D(1)\frac{1}{h(-D)} \sum_{C\in\mathrm{CL}(K)} F_{d,k}(\tau_C)$$

where $\{\tau_C : C \in \mathrm{CL}(K)\}$ is a $\mathrm{CL}(K)$-orbit of Heegner points on $X_0(4d^2)$.

Proof. By the proof of Lemma 3.4, $\tau_{\beta,\mathfrak{a}}$ as a Heegner point on $X_0(4d^2)$ depends only on the ideal class $[\mathfrak{a}]$ (and β). So we can write $\tau_{\beta,\mathfrak{a}} = \tau_{\beta,[\mathfrak{a}]}$. In particular, the Heegner point $\tau_{\beta,C}$ in Theorem 3.5 can be written as $\tau_{\beta,C} = \tau_{\beta,[\mathfrak{a}]}$ with $[\mathfrak{a}] = C^{-1}$.

Observe that $\overline{\psi_{d,k}(\mathfrak{a})} = \psi_{d,k}(\bar{\mathfrak{a}})$ and

$$\psi_{d,k}(\bar{\mathfrak{a}})\psi_{d,k}(\mathfrak{a}) = N(\mathfrak{a})^{2k-1} = a^{2k-1}.$$

Let $\psi_{d,k}$ be a fixed Hecke character in the subfamily $\Psi^{(2)}_{d,k,\beta}$. Then by averaging the central value formula in Theorem 3.2 we obtain

$$\begin{aligned}
\sum_{\psi\in\Psi^{(2)}_{d,k,\beta}} L(\psi, k) &= \sum_{\xi\in\mathrm{CL}^{(2)}(K)^\wedge} L(\psi_{d,k}\xi, k)\\
&= \kappa\#\mathrm{CL}_2(K) \sum_{\xi\in\mathrm{CL}^{(2)}(K)^\wedge}\ \sum_{C_1,C_2\in\mathrm{CL}^{(2)}(K)} \frac{\theta_{d,k-1}(\tau_{\beta,\mathfrak{a}_1^2})\overline{\theta_{d,k-1}(\tau_{\beta,\mathfrak{a}_2^2})}}{\psi_{d,k}\xi(\bar{\mathfrak{a}}_1)\psi_{d,k}\xi(\mathfrak{a}_2)}\\
&= \kappa\#\mathrm{CL}_2(K) \sum_{C_1,C_2\in\mathrm{CL}^{(2)}(K)} \frac{\theta_{d,k-1}(\tau_{\beta,\mathfrak{a}_1^2})\overline{\theta_{d,k-1}(\tau_{\beta,\mathfrak{a}_2^2})}}{\psi_{d,k}(\bar{\mathfrak{a}}_1)\psi_{d,k}(\mathfrak{a}_2)} \sum_{\xi\in\mathrm{CL}^{(2)}(K)^\wedge} \frac{1}{\xi(\bar{\mathfrak{a}}_1\mathfrak{a}_2)}\\
&= \kappa h(-D) \sum_{C\in\mathrm{CL}^{(2)}(K)} \frac{|\theta_{d,k-1}(\tau_{\beta,\mathfrak{a}^2})|^2}{a^{2k-1}},
\end{aligned}$$

where we used

$$\sum_{\xi\in \mathrm{CL}^{(2)}(K)^\wedge} \frac{1}{\xi(\bar{\mathfrak{a}_1}\mathfrak{a}_2)} = \begin{cases} \dfrac{h(-D)}{\#\mathrm{CL}_2(K)}, & \text{if } C_1 = C_2, \\ 0, & \text{if } C_1 \neq C_2. \end{cases}$$

Since

$$\mathrm{Im}(\tau_{\beta,\mathfrak{a}^2}) = \frac{\sqrt{D}}{8\beta d^2 a^2},$$

we have

$$\sum_{\xi\in \mathrm{CL}^{(2)}(K)^\wedge} L(\psi_{d,k}\xi, k) = h(-D)\kappa' \sum_{C\in \mathrm{CL}^{(2)}(K)} F_{d,k}(\tau_{\beta,\mathfrak{a}^2})$$

with

$$\kappa' := \kappa \left(\frac{8\beta d^2}{\sqrt{D}}\right)^{k-\frac12} = \frac{\pi}{\sqrt{D}} \frac{2(8\pi)^{k-1}}{(k-1)!} = \frac{\pi}{\sqrt{D}} c(k).$$

Using the Dirichlet class number formula $L_D(1) = \pi h(-D)/\sqrt{D}$ (recall that $\#\mathcal{O}_K^\times = 2$ since $D > 4$), we obtain

$$\frac{1}{\#\mathrm{CL}^{(2)}(K)} \sum_{\psi\in\Psi^{(2)}_{d,k,\beta}} L(\psi, k) = c(k) L_D(1) \frac{1}{\#\mathrm{CL}^{(2)}(K)} \sum_{C\in \mathrm{CL}^{(2)}(K)} F_{d,k}(\tau_{\beta,\mathfrak{a}^2}).$$

As remarked at the beginning of the proof, we have $\tau_{\beta,\mathfrak{a}^2} = \tau_{\beta,C^{-2}}$ for $\mathfrak{a} \in C^{-1}$. Because $[\mathfrak{a}] \mapsto [\mathfrak{a}^2]$ gives an isomorphism between $\mathrm{CL}^{(2)}(K)$ and $\mathrm{CL}(K)^2$, the proof is complete. $\square$

4. Theta functions of half-integral weight

In this section we use the Maass level-raising operators and the Serre-Stark basis theorem to study properties of the theta function $\theta_{d,k}$ and compute the asymptotic expansion of $F_{d,k}$ in the cusps of $X_0(4d^2)$.

For a half-integer $r \in \frac12 + \mathbb{Z}$, define

$$z^{\frac12} = |z|^{\frac12} e^{\frac{i\theta}{2}} \quad \text{if } z = |z|e^{i\theta}, \quad -\pi < \theta \leq \pi,$$

and $z^r = (z^{\frac12})^{2r}$. Define the multiplier system $\vartheta : \Gamma_0(4) \to \mathbb{C}$ by

$$\vartheta(A) = \varepsilon_\delta \left(\frac{\gamma}{\delta}\right) \quad \text{for} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4)$$

where

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \mod 4, \\ -i & \text{if } d \equiv -1 \mod 4 \end{cases}$$

and $\left(\frac{\gamma}{\delta}\right)$ is the (extended) Jacobi symbol (see [I, (2.73)-(2.76)]).

Let $r \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. Define $M_r^* := M_r^*(\Gamma_0(4d^2), \left(\frac{d}{\cdot}\right))$ to be the space of real-analytic modular forms of weight r, level $4d^2$, and character $(d/\cdot)$. These are the real-analytic functions $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f|_r A = \left(\frac{d}{\delta}\right) \vartheta(A) f \quad \text{for} \quad A \in \Gamma_0(4d^2)$$

where

$$f|_r A(z) = (\gamma z + \delta)^{-r} f(Az)$$

is the usual slash operator.

Note that

$$\theta_{d,0}(z) = \sum_{(n,d)=1} \left(\frac{d}{n}\right) e(n^2 z) \in M_{\frac{1}{2}}^*$$

and

$$\theta_{d,1}(z) = \sum_{(n,d)=1} \left(\frac{d}{n}\right) n e(n^2 z) \in M_{\frac{3}{2}}^*,$$

and that $\theta_{d,0}$ and $\theta_{d,1}$ are holomorphic.

Define the Maass level-raising operator by

$$\partial_r := D - \frac{r}{4\pi y}, \quad D := \frac{1}{2\pi i} \frac{d}{dz}$$

and define its l-th iterate by

$$\partial_r^l := \partial_{r+2l-2} \circ \partial_{r+2l-4} \circ \cdots \circ \partial_{r+2} \circ \partial_r, \quad l \in \mathbb{Z}_{>0}.$$

A straightforward calculation shows that the operator ∂_r commutes with the slash operator $|_r A$,

$$\partial_r(f|_r A) = (\partial_r f)|_{r+2} A, \quad A \in \mathrm{SL}_2(\mathbb{R}). \tag{4.1}$$

It follows that

$$\partial_r^l : M_r^* \to M_{r+2l}^*. \tag{4.2}$$

Note that while the operator ∂_r^l preserves modularity (with an increase in weight), it destroys holomorphicity.

Proposition 4.1. *Let $d \equiv 1 \mod 4$ be squarefree such that* $\mathrm{sign}(d) = (-1)^k$.

(1) *When $k = 2l$ is even, $\theta_{d,k} = \partial_{\frac{1}{2}}^l \theta_{d,0}$.*

(2) *When $k = 2l + 1$ is odd, $\theta_{d,k} = \partial_{\frac{3}{2}}^l \theta_{d,1}$.*

(3) *$\theta_{d,k} \in M_{k+\frac{1}{2}}^*$.*

Proof. Let $r \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. By induction, one can show that

$$\partial_r^l = \sum_{j=0}^{l} \binom{l}{j} \frac{\Gamma(l+r)}{\Gamma(j+r)} \left(\frac{-1}{4\pi y}\right)^{l-j} D^j = \frac{(-1)^l l!}{(4\pi y)^l} \sum_{j=0}^{l} \binom{l+r-1}{l-j} \frac{(-4\pi y)^j}{j!} D^j.$$

In particular, if $f \in M_r^*$ is holomorphic with Fourier expansion at $x = \infty$ of the form

$$f(z) = \sum_{n=0}^{\infty} a(n)e(nz),$$

then

$$\partial_r^l(f) = \frac{(-1)^l l!}{(4\pi y)^l} \sum_{n=0}^{\infty} a(n) L_l^{r-1}(4\pi n y) e(nz) \tag{4.3}$$

where

$$L_l^{\alpha}(z) = \sum_{j=0}^{l} \binom{l+\alpha}{l-j} \frac{(-z)^j}{j!}, \quad \alpha \in \mathbb{C},$$

is the l-th generalized Laguerre polynomial.

(1) After a substitution $j \mapsto l - j$, one can check that

$$L_l^{-\frac{1}{2}}(z) = (-1/4)^l \widetilde{H}_{2l}(\sqrt{z})/l! \tag{4.4}$$

where $\widetilde{H}_k$ is the k-the Hermite polynomial defined in (3.4). So using (3.5) we have

$$\begin{aligned}
\partial_{\frac{1}{2}}^l \theta_{d,0}(z) &= \frac{1}{(16\pi y)^l} \sum_{(d,n)=1} (\frac{d}{n}) \widetilde{H}_{2l}(\sqrt{4\pi n^2 y}) e(n^2 z) \\
&= (2y)^{-l} \sum_{(d,n)=1} (\frac{d}{n}) H_{2l}(n\sqrt{2y}) e(n^2 z) \\
&= \theta_{d,2l}(z),
\end{aligned}$$

as claimed.

(2) Similarly, one has

$$L_l^{\frac{1}{2}}(z) = (-1/4)^l \widetilde{H}_{2l+1}(\sqrt{z})/2\sqrt{z} l!,$$

and arguing as in (1) we obtain

$$\partial_{\frac{3}{2}}^l \theta_{d,1}(z) = \theta_{d,2l+1}(z).$$

(3) This follows from parts (1) and (2) and the map (4.2). □

Define $S_r^* \subset M_r^*$ to be the subset of modular forms with exponential decay in each cusp for $\Gamma_0(4d^2)$. We call these *cusp forms.*

Proposition 4.2. *Let* $d \equiv 1 \mod 4$ *be squarefree with* $\operatorname{sign}(d) = (-1)^k$.

1. *When* k *is odd* ($d < 0$), $\theta_{d,k}$ *is a cusp form.*
2. *When* k *is even,* $\theta_{d,k}$ *is a cusp form if and only if there exists a prime factor of* d *congruent to* $3 \mod 4$.

Proof. (1) It is well-known that $\theta_{d,1}$ is a holomorphic cusp form. Thus $\theta_{d,1}$ has exponential decay in each cusp x for $\Gamma_0(4d^2)$ which is singular with respect to the multiplier system ϑ (in the sense of the definition on p. 44 of [I]). Because $\theta_{d,k} = \partial^l_{3/2}\theta_{d,1}$ with $l = (k-1)/2$ (Proposition 4.1, part (1)), is suffices to show that $\partial^l_{3/2}$ preserves the property of having exponential decay in the singular cusps for $\Gamma_0(4d^2)$. This can be verified by a direct calculation applying $\partial^l_{3/2}$ to the Fourier expansions of $\theta_{d,1}$ in the singular cusps (see Proposition 4.4, part (2)).

(2) By a theorem of Serre and Stark [SS], the cuspidality of $\theta_{d,0}$ is determined by the local decomposition of $(d/\cdot)$. Indeed, let $\chi = (d/\cdot) = \prod_{p|d}\chi_p$ be the local decomposition of χ where

$$\chi_p = \begin{cases} (\frac{-p}{\cdot}) & \text{if } p \equiv 3 \mod 4, \\ (\frac{p}{\cdot}) & \text{if } p \equiv 1 \mod 4. \end{cases}$$

Then χ is *totally even* (in the sense of Serre and Stark) if and only if every prime factor p of d satisfies $p \equiv 1 \mod 4$. Therefore, by [SS, Theorem B] $\theta_{d,0}$ is a cusp form if and only if χ is *not* totally even if and only if there is a prime factor p of d congruent to $3 \mod 4$. Now, because $\theta_{d,k} = \partial^l_{1/2}\theta_{d,0}$ with $l = k/2$ (Proposition 4.1, part (2)), for $k > 0$ (even) the argument in (1) implies that $\theta_{d,k}$ is a cusp form if $\theta_{d,0}$ is a cusp form. On the other hand, if $\theta_{d,0}$ is not a cusp form, (the proof of) Proposition 4.4 implies that $\theta_{d,k}$ is not a cusp form. In fact, Proposition 4.4 gives a concrete proof that the cuspidality of $\theta_{d,k}$ is the same as that of $\theta_{d,0}$ or $\theta_{d,1}$ depending on whether k is even or odd. □

Proposition 4.3. *Assume that* $(-1)^{k-1} = \mathrm{sign}(d)$. *Then the theta function* $\theta_{d,k-1}$ *is a real analytic modular form of weight* $k - \frac{1}{2}$ *for* $\Gamma_0(4d^2)$ *with character* $(d/\cdot)$. *Moreover,*

(1) $\theta_{d,0}$ *and* $\theta_{d,1}$ *are holomorphic.*
(2) $\theta_{d,k-1}$ *is a cusp form if and only if either* $d > 0$ *and there exists a prime factor of* d *congruent to* $3 \bmod 4$, *or* $d < 0$.

Proof. This follows from Propositions 4.1 and 4.2. □

We are now in a position to compute the asymptotic expansion of $F_{d,k}$ in the cusps of $X_0(4d^2)$.

Proposition 4.4. *(1) Suppose that* $k \geq 1$ *is odd. Then in each cusp* x *for* $\Gamma_0(4d^2)$ *which is singular with respect to the multiplier system* ϑ *we have*

$$F_{d,k}(\sigma_x z) = y^{\frac{1}{2}} \frac{(k-1)!^2}{(16\pi)^{k-1}(\frac{k-1}{2})!^2}|a_x(0)|^2 + O(e^{-cy}) \quad \text{as} \quad y \to \infty$$

for some $c > 0$. *Here* $\sigma_x \in \mathrm{SL}_2(\mathbb{R})$ *is a scaling matrix such that* $\sigma_x(\infty) = x$, *and* $a_x(0)$ *is the zeroth Fourier coefficient in the Fourier expansion of* $\theta_{d,0}$ *at* x.

Moreover, $F_{d,k}$ *has exponential decay in each cusp* x *for* $\Gamma_0(4d^2)$ *which is nonsingular with respect to the multiplier system* ϑ.

(2) Suppose that $k \geq 1$ is even. Then $F_{d,k}$ has exponential decay in each cusp x for $\Gamma_0(4d^2)$.

Proof. (1) Suppose that $k \geq 1$ is odd. By definition of $F_{d,k}$ we have

$$F_{d,k}(\sigma_x z) = \mathrm{Im}(\sigma_x z)^{k-\frac{1}{2}} |\theta_{d,k-1}(\sigma_x z)|^2.$$

Note that if $\sigma_x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$\mathrm{Im}(\sigma_x z)^{k-\frac{1}{2}} = \frac{\mathrm{Im}(z)^{k-\frac{1}{2}}}{|cz+d|^{2(k-\frac{1}{2})}}.$$

By Proposition 4.1 and (4.1),

$$\theta_{d,k-1}(\sigma_x z) = (cz+d)^{k-\frac{1}{2}} \theta_{d,k-1}|_{k-\frac{1}{2}} \sigma_x(z) = (cz+d)^{k-\frac{1}{2}} \partial_{\frac{1}{2}}^{\frac{k-1}{2}} \theta_{d,0}|_{\frac{1}{2}} \sigma_x(z).$$

Suppose that x is singular with respect to the multiplier system ϑ. Let

$$\theta_{d,0}|_{1/2}\sigma_x(z) = \sum_{n=0}^{\infty} a_x(n) e(nz)$$

be the Fourier expansion of the holomorphic modular form $\theta_{d,0}$ at the singular cusp x. By (4.3) and (4.4), we have

$$\begin{aligned} \partial_{1/2}^{\frac{k-1}{2}} (\theta_{d,0}|_{1/2}\sigma_x)(z) &= \frac{(-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2}\right)!}{(4\pi y)^{\frac{k-1}{2}}} \sum_{n=0}^{\infty} a_x(n) L_{\frac{k-1}{2}}^{-1/2}(4\pi n y) e(nz) \\ &= \frac{1}{(16\pi y)^{\frac{k-1}{2}}} \sum_{n=0}^{\infty} a_x(n) \widetilde{H}_{k-1}(2\sqrt{\pi n y}) e(nz). \end{aligned}$$

Putting things together, we have

$$\begin{aligned} F_{d,k}(\sigma_x z) &= \frac{\mathrm{Im}(z)^{k-\frac{1}{2}}}{|cz+d|^{2(k-\frac{1}{2})}} \frac{1}{(16\pi y)^{k-1}} |cz+d|^{2(k-\frac{1}{2})} \left| \sum_{n=0}^{\infty} a_x(n) \widetilde{H}_{k-1}(2\sqrt{\pi n y}) e(nz) \right|^2 \\ &= y^{1/2} \frac{1}{(16\pi)^{k-1}} \left| \sum_{n=0}^{\infty} a_x(n) \widetilde{H}_{k-1}(2\sqrt{\pi n y}) e(nz) \right|^2 \\ &= y^{1/2} \frac{\widetilde{H}_{k-1}(0)^2}{(16\pi)^{k-1}} |a_x(0)|^2 + O_x(e^{-cy}) \end{aligned}$$

as $y \to \infty$ for some explicit $c > 0$. Finally, using the definition of the Hermite polynomial $\widetilde{H}_{k-1}(x)$ and that $k \geq 1$ is odd, we compute

$$\widetilde{H}_{k-1}(0)^2 = \frac{(k-1)!^2}{\left(\frac{k-1}{2}\right)!^2},$$

so that

$$F_{d,k}(\sigma_x z) = y^{1/2} \frac{(k-1)!^2}{(16\pi)^{k-1} \left(\frac{k-1}{2}\right)!^2} |a_x(0)|^2 + O_x(e^{-cy})$$

as $y \to \infty$.

Suppose that x is nonsingular with respect to the multiplier system ϑ. For such a cusp one has the Fourier expansion (see [I, (2.63)-(2.64)])

$$\theta_{d,0}|_{1/2}\sigma_x(z) = e(\kappa_x z)\sum_{n=0}^{\infty} a_x(n)e(nz)$$

with $0 < \kappa_x < 1$ such that $\vartheta(\sigma_x) = e(\kappa_x)$. Then an argument similar to the one just given shows that $F_{d,k}$ has exponential decay at x.

(2) Suppose that $k \geq 1$ is even. The proof in this case is similar except that the zeroth Fourier coefficient in the Fourier expansion of the holomorphic cusp form $\theta_{d,1}$ at every singular cusp is zero. □

5. Galois suborbits of Heegner points

Let N be a positive integer such that every prime factor of N is split or ramified in $K = \mathbb{Q}(\sqrt{-D})$. Let $\Lambda_D(N)$ be the set of Heegner points of discriminant $-D$ on $X_0(N)$. The set $\Lambda_D(N)$ is divided into $2^{t(N)}$ simple, transitive $\mathrm{Gal}(H/K)$-orbits of size $h(-D)$ where $t(N)$ is the number of prime divisors of N. By Siegel's theorem, one knows that $\#\Lambda_D(N) \to \infty$ as $D \to \infty$. Therefore, given a subgroup $G \subset G_H := \mathrm{Gal}(H/K)$ and $\tau \in \Lambda_D(N)$, it is natural to ask how the Galois (sub)orbit $G \cdot \tau = \{\tau^\sigma : \sigma \in G\}$ is distributed on $X_0(N)$ as $D \to \infty$.

In [D], Duke proved that the full Galois orbit $G_H \cdot \tau$ is equidistributed on $X_0(1)$ as $D \to \infty$. More recently, Harcos and Michel [HM, Theorem 6] proved that if the index $|G_H : G|$ is bounded by a sufficiently small positive power of D, the Galois suborbit $G \cdot \tau$ is equidistributed on $X_0(N)$ as $D \to \infty$. This theorem rests on a deep subconvexity bound for Rankin-Selberg L–functions proved by Harcos and Michel in [HM], and period relations of Waldspurger [Wal] and Zhang [Zh, Zh2] for hyperbolic Weyl sums associated to Maass cusp forms.

The are many situations in number theory where one would like to use the equidistribution of Heegner points to obtain information about the growth of a certain quantity, but the "test functions" involved are not compactly supported, and in fact, grow in the cusps of $X_0(N)$ as $y \to \infty$. One example is the function $F_{d,k}$ appearing in the average formula (2.2). Other examples occur in the work of Duke [D2] on traces of singular moduli, and the work of Folsom and the second author on the limiting distribution of traces of Maass-Poincaré series [FM], and the asymptotic distribution of the partition function [FM2].

In this section we prove an "equidistribution theorem" for test functions satisfying a moderate growth condition in the cusps of $X_0(N)$. The growth condition is defined as follows: Let $F : \mathbb{H} \to \mathbb{C}$ be a C^∞, $\Gamma_0(N)$-invariant function. We say that F has *cuspidal growth of power* α for some $\alpha \in \mathbb{R}$ if for every cusp x of $X_0(N)$ there exists a constant $c_x \in \mathbb{C}$ (possibly equal to 0) such that for each integer $a \in \mathbb{Z}_{\geq 0}$,

$$\Delta^a(F(\sigma_x z) - c_x y^\alpha) = O(e^{-cy}) \quad \text{as} \quad y = \mathrm{Im}(z) \to \infty$$

for some $c = c_x(a) > 0$. Here $\Delta = -y^2(\partial_x^2 + \partial_y^2)$ is the hyperbolic Laplacian where Δ^a means we apply the Laplacian a-times, and $\sigma_x \in \mathrm{SL}_2(\mathbb{R})$ is a scaling matrix such that $\sigma_x(\infty) = x$.

Theorem 5.1. *Suppose that $F : \mathbb{H} \to \mathbb{C}$ is a C^∞, $\Gamma_0(N)$-invariant function with cuspidal growth of power $\alpha < 1$. Then for all $\epsilon > 0$ and any fixed Heegner point τ of discriminant $-D$,*

$$\frac{1}{\#G}\sum_{\sigma\in G} F(\tau^\sigma) = \int_{Y_0(N)} F(z)d\mu(z) \tag{5.1}$$
$$+ O(|G_H : G|D^{-\frac{1}{2827}}) + O(|G_H : G|\, D^{-\delta_\epsilon(\alpha)}) + O(D^{-\frac{(1-\alpha)}{2}})$$

as $D \to \infty$ where

$$\delta_\epsilon(\alpha) := \begin{cases} \frac{1}{1889} - \epsilon, & \text{if} \quad \alpha \le \frac{1}{2} \\ \frac{1}{1889} - \alpha(\alpha - \frac{1}{2}) - \epsilon, & \text{if} \quad \frac{1}{2} < \alpha < \frac{1}{4} + \frac{1}{2}\sqrt{\frac{1}{4} + 4(\frac{1}{1889} - \epsilon)}. \end{cases}$$

Moreover, assuming GRH the estimate (5.1) holds with $\delta_\epsilon(\alpha)$ replaced by

$$\delta_{1,\epsilon}(\alpha) := \begin{cases} \frac{1}{2} - \epsilon, & \text{if} \quad \alpha \le \frac{1}{2} \\ \frac{1}{2} - \alpha(\alpha - \frac{1}{2}) - \epsilon, & \text{if} \quad \frac{1}{2} < \alpha < \frac{1}{4} + \frac{1}{2}\sqrt{\frac{1}{4} + 4(\frac{1}{2} - \epsilon)}. \end{cases}$$

Remark 5.2. In the special case $G = \mathrm{CL}(K)^2$ the constants 1/2827 and 1/1889 in Theorem 5.1 can be replaced by any constant less than 1/8. This is because for a *square* suborbit, the class group characters appearing in the hyperbolic Weyl sums in [HM, section 6.4] are genus characters, which allows one to factor the L–functions which arise in the period relations for these hyperbolic Weyl sums and apply a subconvexity bound of Blomer and Harcos [BH, Theorem 2] for degree 2 L-functions. This accounts for the presence of the exponent $\delta < 1/8$ in the error terms in Theorem 1.1.

Proof of Theorem 5.1. We begin by constructing a C^∞, $\Gamma_0(N)$-invariant function with growth coinciding precisely with that of F in the cusps x of $X_0(N)$, and which vanishes on the Heegner points $\Lambda_D(N)$.

Lemma 5.3. *Let $T > 1$, $\alpha > 0$, and $c_x \in \mathbb{C}$ for cusps x of $X_0(N)$. Then there exists a C^∞, $\Gamma_0(N)$-invariant function $\eta_T : \mathbb{H} \to \mathbb{C}$ such that*

$$\eta_T(\sigma_x z) = \begin{cases} 0, & 1 < y < T \\ c_x y^\alpha \chi(y/T), & T \le y \le 2T \\ c_x y^\alpha, & y > 2T \end{cases}$$

where $\chi : \mathbb{R}^+ \to [0, 1]$ is a C^∞ function such that

$$\chi(t) = \begin{cases} 0, & t < 1 \\ 1, & t > 2. \end{cases}$$

Proof. Define η_T by (see [I2, (3.10)])

$$\eta_T(z) := \sum_x c_x \sum_{\gamma \in \Gamma_x \backslash \Gamma_0(N)} \psi_T(\mathrm{Im}(\sigma_x^{-1}\gamma z))$$

where $\psi_T \in C_0^\infty(\mathbb{R}^+)$ is defined by

$$\psi_T(t) := t^\alpha \chi(\frac{t}{T}).$$

Then [I2, (3.17)] with $m = 0$ combined with the fact

$$\min \left\{ c > 0 : \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \sigma_{x_1}^{-1}\Gamma_0(N)\sigma_{x_2} \right\} \geq 1 \tag{5.2}$$

for all cusps x_1, x_2 of $X_0(N)$ (see [I2, (2.28)-(2.31)]) shows that η_T has the properties stated in the lemma. $\square$

Lemma 5.4. *For $T \gg \sqrt{D}$, the function η_T vanishes on the Heegner points $\Lambda_D(N)$.*

Proof. Recall that a Heegner point $\tau \in \Lambda_D(N)$ has the form

$$\tau = \tau_{[\mathfrak{A}]} = \frac{b + \sqrt{-D}}{2Na}$$

where $\mathfrak{A}$ is a primitive ideal and $a = N_{K/\mathbb{Q}}(\mathfrak{A})$.

For $\gamma \in \Gamma_0(N)$ write

$$\sigma_x^{-1}\gamma = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Then

$$\mathrm{Im}(\sigma_x^{-1}\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2} \leq \frac{1}{|c\tau + d|^2}\frac{\sqrt{D}}{2N}.$$

Assume that $c = 0$. Then $d = 1$ (see [I2, (2.15)-(2.17)]), and we have

$$\mathrm{Im}(\sigma_x^{-1}\gamma\tau) \leq \frac{\sqrt{D}}{2N}. \tag{5.3}$$

Next, assume that $c \neq 0$. Write

$$c\tau + d = \left(\frac{bc}{2Na} + d\right) + i\left(\frac{\sqrt{D}c}{2Na}\right),$$

so that

$$|c\tau + d|^2 = \left(\frac{bc}{2Na} + d\right)^2 + \left(\frac{\sqrt{D}c}{2Na}\right)^2 \geq \frac{c^2 D}{4N^2a^2}.$$

By Minkowski's theorem, every class $[\mathfrak{A}]$ contains an ideal $\mathfrak{A}$ of norm

$$N_{K/\mathbb{Q}}(\mathfrak{A}) = a < \frac{2}{\pi}\sqrt{D}.$$

Then

$$\frac{c^2D}{4N^2a^2} > \frac{c^2D}{4N^2}\frac{\pi^2}{4D} = \frac{c^2\pi^2}{16N^2},$$

so that

$$\mathrm{Im}(\sigma_x^{-1}\gamma\tau) \leq \frac{1}{|c\tau+d|^2}\frac{\sqrt{D}}{2N} < \frac{16N^2}{c^2\pi^2}\frac{\sqrt{D}}{2N} \leq \frac{8N}{\pi^2}\sqrt{D} \tag{5.4}$$

where for the last inequality we used (5.2).

Since $\psi_T(y) = 0$ for $y < T$, we see from the inequalities (5.3) and (5.4) that for $T \gg \sqrt{D}$,

$$\psi_T(\mathrm{Im}(\sigma_x^{-1}\gamma\tau)) = 0$$

for all $\gamma \in \Gamma_x\backslash\Gamma_0(N)$. It follows from the definition of η_T that $\eta_T(\tau) = 0$. □

Define the "regularized" function

$$F_T(z) := F(z) - \eta_T(z).$$

In light of the preceding lemma, we see that to prove Theorem 5.1 it suffices to prove the following proposition.

Proposition 5.5. *Let the notation be as in Theorem 5.1. We have*

$$\frac{1}{\#G}\sum_{\sigma\in G} F_T(\tau^\sigma) = \int_{Y_0(N)} F(z)d\mu(z) \\ + O(|G_H : G|D^{-\frac{1}{2827}}) + O(|G_H : G|\, D^{-\delta_{2,\epsilon}(\alpha)}) + O(D^{-\frac{(1-\alpha)}{2}})$$

as $T \gg \sqrt{D}$ and $D \to \infty$. Here $\delta_{2,\epsilon}(\alpha)$ equals $\delta_\epsilon(\alpha)$ in general, and $\delta_{1,\epsilon}(\alpha)$ if we assume GRH.

Proof. Let $T \geq T_0 \gg 1$ where T_0 is a fixed cutoff parameter which is independent of D. We introduce T_0 in order to decompose F_T into a sum of two functions so that we can isolate the contribution of η_T to the spectral decomposition.

Consider the decomposition

$$F_T(z) = F_{T_0}(z) + \tilde{\eta}_T(z)$$

where

$$\tilde{\eta}_T(z) := \eta_{T_0}(z) - \eta_T(z).$$

By the properties of η_{T_0} given in Lemma 5.3 and our assumption that F has cuspidal growth of power $\alpha < 1$, we have

$$\Delta^a F_{T_0}(\sigma_x z) = O(e^{-cy}) \quad \text{as} \quad y \to \infty$$

for each integer $a \in \mathbb{Z}_{\geq 0}$. Therefore the proof of [HM, Theorem 6], combined with subconvexity bounds of Blomer, Harcos, and Michel [BHM] and Blomer and Harcos

[BH, Corollary 1], implies that

$$\frac{1}{\#G}\sum_{\sigma\in G} F_{T_0}(\tau^\sigma) = \int_{Y_0(N)} F_{T_0}(z)d\mu(z) + O(|G_H : G| D^{-\frac{1}{2827}})$$

as $D \to \infty$.

To complete the proof it suffices to show that

$$\frac{1}{\#G}\sum_{\sigma\in G} \tilde{\eta}_T(\tau^\sigma) = \int_{Y_0(N)} \eta_{T_0}(z)d\mu(z) + O(|G_H : G|\, D^{-\delta_2(\alpha)}) + O(D^{-\frac{(1-\alpha)}{2}})$$

as $T \gg \sqrt{D}$ and $D \to \infty$.

By combining the explicit construction of $\tilde{\eta}_T$ in Lemma 5.3 with [I2, (7.12)], [I2, Theorem 11.3] and [I2, (7.13)], one has

$$\tilde{\eta}_T(z) = \langle \tilde{\eta}_T, 1\rangle_2 + \frac{1}{2\pi}\sum_x c_x \int_{\mathbb{R}} \left(\hat{\psi}_{T_0}(\frac{1}{2}+it) - \hat{\psi}_T(\frac{1}{2}+it)\right) E_x(z, \frac{1}{2}+it)dt \quad (5.5)$$

where (see [I2, (3.13)])

$$\hat{\psi}(s) := \int_0^\infty \psi(y) y^{-(s+1)} dy$$

and $E_x(z,s)$ is the real-analytic Eisenstein series associated to the cusp x.

Averaging (5.5) over the suborbit $G\cdot\tau$ yields

$$\begin{aligned}\frac{1}{\#G}\sum_{\sigma\in G} \tilde{\eta}_T(\tau^\sigma) &= \int_{Y_0(N)} \eta_{T_0}(z)d\mu(z) - \int_{Y_0(N)} \eta_T(z)d\mu(z) \\ &\quad + \frac{1}{2\pi}\sum_x c_x \int_{\mathbb{R}} \left(\hat{\psi}_{T_0}(\frac{1}{2}+it) - \hat{\psi}_T(\frac{1}{2}+it)\right) W_x(t)dt\end{aligned}$$

where

$$W_x(t) := \frac{1}{\#G}\sum_{\sigma\in G} E_x(\tau^\sigma, \frac{1}{2}+it).$$

Using Fourier analysis one obtains the decomposition

$$W_x(t) = \frac{1}{h(-D)} \sum_{\substack{\xi\in\widehat{G_H} \\ \xi_{|G}=1}} \sum_{\sigma\in G_H} \xi(\sigma) E_x(\tau^\sigma, \frac{1}{2}+it).$$

The argument in [HM, pp. 648-649] can be generalized to modular curves of *non-squarefree* level, and the estimate of $W_x(t)$ can be reduced to an analogous estimate for

$$\sum_{\sigma\in G_H} \xi(\sigma) E(\tau^\sigma, \frac{1}{2}+it)$$

where $E(z,s)$ is the *full level* Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ and $\{\tau^\sigma\}$ is the set of Heegner points of discriminant $-D$ on the modular curve $X_0(1)$.[1] By a classical

[1] We thank Philippe Michel for explaining this fact to us.

formula of Dirichlet one has an identity of the form (see [GZ, p. 248])

$$\left|\sum_{\sigma\in G}\xi(\sigma)E(\tau^{\sigma},\frac{1}{2}+it)\right|^2=\frac{\sqrt{D}}{2}\left|L(\Theta_{\xi},\frac{1}{2}+it)\right|^2$$

where Θ_{ξ} is the theta function associated to the class group character ξ. By Blomer, Harcos, and Michel [BHM] one has the following deep subconvexity bound,

$$L(\Theta_{\xi},\frac{1}{2}+it)\ll(1+|t|)^{A_1}D^{\frac{1}{4}-\frac{1}{1889}}$$

for some $A_1>0$. By combining the preceding facts with Siegel's theorem

$$h(-D)\gg_{\epsilon}D^{\frac{1}{2}-\epsilon}$$

we obtain the estimate

$$W_x(t)\ll(1+|t|)^{A_2}\frac{|G_H:G|}{h(-D)}D^{\frac{1}{2}-\frac{1}{1889}}\ll_{\epsilon}(1+|t|)^{A_2}\,|G_H:G|\,D^{-\frac{1}{1889}+\epsilon} \tag{5.6}$$

for some $A_2>0$

By Lemma 5.6, for all $B>0$ we have the estimate

$$\int_{\mathbb{R}}\left|\hat{\psi}_{T_0}(\frac{1}{2}+it)-\hat{\psi}_T(\frac{1}{2}+it)\right|(1+|t|)^B dt\ll C(\alpha,T) \tag{5.7}$$

where

$$C(\alpha,T):=\begin{cases}\log(T), & \alpha\le\frac{1}{2}\\ T^{\alpha-\frac{1}{2}}, & \alpha>\frac{1}{2}.\end{cases}$$

If $\alpha\le1/2$, take $\sqrt{D}\ll T\ll D$ and combine (5.6) and (5.7) to obtain

$$\frac{1}{2\pi}\sum_x c_x\int_{\mathbb{R}}\left(\hat{\psi}_{T_0}(\frac{1}{2}+it)-\hat{\psi}_T(\frac{1}{2}+it)\right)W_x(t)dt=O\left(|G_H:G|\,\frac{\log(D)}{D^{\frac{1}{1889}-\epsilon}}\right).$$

Similarly, if $\alpha>1/2$ take $\sqrt{D}\ll T\ll D^{\alpha}$ with $\alpha<\frac{1}{4}+\frac{1}{2}\sqrt{\frac{1}{4}+4(\frac{1}{1889}-\epsilon)}$ to obtain

$$\frac{1}{2\pi}\sum_x c_x\int_{\mathbb{R}}\left(\hat{\psi}_{T_0}(\frac{1}{2}+it)-\hat{\psi}_T(\frac{1}{2}+it)\right)W_x(t)dt=O\left(|G_H:G|\,\frac{1}{D^{\frac{1}{1889}-\alpha(\alpha-\frac{1}{2})-\epsilon}}\right).$$

We conclude that

$$\frac{1}{\#G}\sum_{\sigma\in G}\tilde{\eta}_T(\tau^{\sigma})=\int_{Y_0(N)}\eta_{T_0}(z)d\mu(z)-\int_{Y_0(N)}\eta_T(z)d\mu(z)+O\left(|G_H:G|\,D^{-\delta_{\epsilon}(\alpha)}\right)$$

where

$$\delta_{\epsilon}(\alpha):=\begin{cases}\frac{1}{1889}-\epsilon, & \alpha\le\frac{1}{2}\\ \frac{1}{1889}-\alpha(\alpha-\frac{1}{2})-\epsilon, & \frac{1}{2}<\alpha<\frac{1}{4}+\frac{1}{2}\sqrt{\frac{1}{4}+4(\frac{1}{1889}-\epsilon)}.\end{cases}$$

Assuming GRH, we can replace 1/1889 with 1/2 in the Blomer-Harcos-Michel subconvexity bound, and we get the same estimate with $\delta_\epsilon(\alpha)$ replaced by

$$\delta_{1,\epsilon}(\alpha) := \begin{cases} \frac{1}{2} - \epsilon, & \alpha \le \frac{1}{2} \\ \frac{1}{2} - \alpha(\alpha - \frac{1}{2}) - \epsilon, & \frac{1}{2} < \alpha < \frac{1}{4} + \frac{1}{2}\sqrt{\frac{1}{4} + 4(\frac{1}{2} - \epsilon)}. \end{cases}$$

Finally, if $T \gg \sqrt{D}$ a straightforward estimate yields

$$\int_{Y_0(N)} \eta_T(z) d\mu(z) = O(D^{-\frac{(1-\alpha)}{2}}).$$

□

It remains to prove Lemma 5.6.

Lemma 5.6. *For all $B > 0$,*

$$\int_{\mathbb{R}} \left| \hat{\psi}_{T_0}(\frac{1}{2} + it) - \hat{\psi}_T(\frac{1}{2} + it) \right| (1 + |t|)^B dt \ll C(\alpha, T)$$

where

$$C(\alpha, T) := \begin{cases} \log(T), & \alpha \le \frac{1}{2} \\ T^{\alpha - \frac{1}{2}}, & \alpha > \frac{1}{2}. \end{cases}$$

Proof. Because $\chi(y/T_0) - \chi(y/T)$ is supported in $(T_0, 2T)$ we have the identity

$$f_T(t) := \hat{\psi}_{T_0}(\frac{1}{2} - it) - \hat{\psi}_T(\frac{1}{2} - it) = \int_{T_0}^{2T} (\chi(y/T_0) - \chi(y/T))\, y^{it + \alpha - \frac{3}{2}} dy.$$

Then integrating by parts k-times and using that the k-th derivative $\chi^{(k)}(y)$ of $\chi(y)$ is supported in $(1, 2)$ yields

$$(-1)^k \prod_{j=0}^{k-1} \left(it + \frac{1}{2} + (\alpha - 1) + j \right) f_T(t) = \left(T_0^{it + \frac{1}{2} + (\alpha-1)} - T^{it + \frac{1}{2} + (\alpha - 1)} \right) \int_1^2 \chi^{(k)}(y) y^{it + \alpha - \frac{3}{2} + k} dy.$$

Suppose that $|t| \ge 1$. Then we have the estimate

$$\left| (-1)^k \prod_{j=0}^{k-1} (it + \frac{1}{2} + (\alpha - 1) + j) \right| \le \prod_{j=0}^{k-1} \left(\frac{1}{2} + (1 - \alpha) + j + 1 \right) |t|^k .$$

Furthermore, we have the estimate

$$\left| T_0^{it + \frac{1}{2} + (\alpha - 1)} - T^{it + \frac{1}{2} + (\alpha - 1)} \right| \le C_1(\alpha, T)$$

where

$$C_1(\alpha, T) := \begin{cases} \frac{2}{T_0^{\frac{1}{2} - \alpha}}, & \alpha \le \frac{1}{2} \\ 2T^{\alpha - \frac{1}{2}}, & \alpha > \frac{1}{2}. \end{cases}$$

Then by combining the preceding facts we obtain

$$|f_T(t)| \leq C_1(\alpha, T)\frac{\max_{1\leq y\leq 2}\left|\chi^{(k)}(y)\right|}{\prod_{j=0}^{k-1}\left(\frac{1}{2}+(1-\alpha)+j+1\right)}\frac{|2^{k+\alpha-\frac{1}{2}}|}{\left|k+\alpha-\frac{1}{2}\right|}\,|t|^{-k}.$$

Because $B > 0$ is fixed and $k \geq 1$ is arbitrary, it follows that

$$\int_{|t|\geq 1} |f_T(t)|\,(1+|t|)^B dt \ll C_1(\alpha, T).$$

Next suppose that $|t| < 1$. For $\alpha \leq 1/2$ we have the estimate

$$|f_T(t)| \leq 2\sup_{y\in\mathbb{R}^+}|\chi(y)|\int_{T_0}^{2T} y^{-1}dy \ll \log(T),$$

and for $\alpha > 1/2$ we have the estimate

$$|f_T(t)| \leq 2\sup_{y\in\mathbb{R}^+}|\chi(y)|\int_{T_0}^{2T} y^{\alpha-\frac{3}{2}}dy \ll T^{\alpha-\frac{1}{2}}.$$

Then because $(1+|t|)^B \ll 1$ for $|t| < 1$, it follows that

$$\int_{|t|<1} |f_T(t)|\,(1+|t|)^B dt \ll C(\alpha, T).$$

$\square$

6. Proof of Theorem 1.1

The average formula (3.6) yields

$$\frac{1}{\#\mathrm{CL}^{(2)}(K)}\sum_{\psi_{d,k}\in\Psi_{d,k}^{(2)}} L(\psi_{d,k}, k) = c(k)L_D(1)\frac{1}{\#\mathrm{CL}(K)^2}\sum_{C\in\mathrm{CL}(K)^2} F_{d,k}(\tau_C)$$

where $\{\tau_C : C \in \mathrm{CL}(K)^2\}$ is a $\mathrm{CL}(K)^2$-orbit of Heegner points on $X_0(4d^2)$.

By Proposition 4.4 (and its proof) we know that $F_{d,k}$ has cuspidal growth of power $\alpha = 1/2$. Let $G_H = \mathrm{CL}(K)$, $G = \mathrm{CL}(K)^2$, $F = F_{d,k}$ and $\alpha = 1/2$ in Theorem 5.1. Since

$$L_D(1) \ll_\epsilon D^\epsilon,$$

and

$$\left|\mathrm{CL}(K) : \mathrm{CL}(K)^2\right| \ll_\epsilon D^\epsilon$$

by genus theory, we obtain the asymptotic formula

$$\frac{1}{\#\mathrm{CL}^{(2)}(K)}\sum_{\psi_{d,k}\in\Psi_{d,k}^{(2)}} L(\psi_{d,k}, k) = c(k)L_D(1)\langle\theta_{d,k-1}, \theta_{d,k-1}\rangle_{\mathrm{Pet}} + O_{d,k,\epsilon}(D^{-\frac{1}{8}+\epsilon})$$

as $D \to \infty$. The appearance of the exponent $1/8$ is justified in Remark 5.2. This proves (1.1). A similar argument can be used to prove (1.2).

7. Proof of Theorem 1.4

Theorem 1.1 implies that

$$\frac{1}{\#\mathrm{CL}^{(2)}(K)} \sum_{\psi_{d,k} \in \Psi^{(2)}_{d,k}} \frac{L(\psi_{d,k}, k)}{L_D(1)} = c(k)\langle \theta_{d,k-1}, \theta_{d,k-1} \rangle_{\mathrm{Pet}} + o(1)$$

as $D \to \infty$. Using $\mathrm{CL}^{(2)}(K) = \mathrm{CL}(K)/\mathrm{CL}_2(K)$, the Dirichlet class number formula, Siegel's theorem and genus theory, we obtain for every $\epsilon > 0$,

$$\begin{aligned} \sum_{\psi_{d,k} \in \Psi^{(2)}_{d,k}} L(\psi_{d,k}, k) &= c(k)\pi \frac{h(-D)^2}{\#\mathrm{CL}_2(K)\sqrt{D}} \left(\langle \theta_{d,k-1}, \theta_{d,k-1} \rangle_{\mathrm{Pet}} + o(1) \right) \\ &\gg_{d,k,\epsilon} D^{\frac{1}{2}-\epsilon}. \end{aligned} \tag{7.1}$$

By Duke, Friedlander, and Iwaniec [DFI] one has the subconvexity bound

$$L(\psi, k) \ll_{d,\epsilon} D^{\frac{1}{2}-\frac{1}{60}+\epsilon} \tag{7.2}$$

for every $\psi \in \Psi_{d,k}$ (here ψ corresponds to a CM cuspidal eigenform for $\Gamma_0(d^2D^2)$ of weight $2k$ with trivial nebentypus). It follows from (7.1) and (7.2) that

$$\#\{\psi_{d,k} \in \Psi^{(2)}_{d,k} : L(\psi_{d,k}, k) \neq 0\} \gg_{d,k,\epsilon} D^{\frac{1}{60}-\epsilon}.$$

8. Proof of Theorem 1.5

We will prove Theorem 1.5 using the main conjecture of Iwasawa theory for imaginary quadratic fields due to Rubin [Ru2]. To deduce Theorem 1.5 from the main conjecture, we must use Iwasawa theoretic techniques a number of times. In particular, we make use of the crucial fact that a certain ideal class group over some $\mathbb{Z}_p$-extension has no non-trivial finite submodule. This type of result is used frequently in Iwasawa theory (see for example [Gre]).

The hypothesis on p in the statement arises in the following way. Assume p splits in K. If $\mathfrak{p}$ is any prime of K above p, then the local condition (defined by Bloch and Kato) at $\mathfrak{p}$ is essentially the *relaxed* local condition and the local condition at $\bar{\mathfrak{p}}$ is the *strict* local condition. In turn, this enables us to use the Selmer groups over the $\mathbb{Z}_p$-extensions more freely.

It is possible that one might obtain a result similar to Theorem 1.5 when p is inert in K. For example, Han [Han] obtained a result similar to Theorem 1.5 for both primes which are split and primes which are inert for imaginary quadratic fields K of class number 1 by using p-adic Hodge theory instead of the formal group theory of elliptic curves. In the future, it might be worthwhile to prove our result for inert primes by adapting Han's techniques.

First we define a Selmer group for a p-adic representation following Bloch and Kato ([BK]).

Definition 8.1 (Bloch-Kato Selmer group)**.** *Let F be a finite extension of $\mathbb{Q}_p$ and O be its ring of integers. Let T be a free O-module of finite rank with $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$-action and V be $T \otimes F$ and A be V/T. For any prime v not above p we let*

$$H^1_f(K_v, V) = \ker\left(H^1(K_v, V) \to H^1(K_v^{un}, V)\right)$$

and for any prime v above p we let

$$H^1_f(K_v, V) = \ker\left(H^1(K_v, V) \to H^1(K_v, V \otimes_F B_{cris})\right)$$

for Fontaine's ring B_{cris}. (The definition of B_{cris} is quite lengthy and we refer the reader to [F]*.) We define the local condition*

$$H^1_f(K_v, A) = \mathrm{im}\left(H^1_f(K_v, V) \to H^1(K_v, A)\right)$$

for every prime v and define the Bloch-Kato p-Selmer group for A by

$$\mathrm{Sel}_p(A/K) := \ker\left(H^1(K, A) \to \prod_v H^1(K_v, A)/H^1_f(K_v, A)\right).$$

Also, the following *relaxed* Selmer group will be useful in our argument.

Definition 8.2. *Let L be any extension of K. We define*

$$S(A/L) := \ker\left(H^1(L, A) \to \prod_{v \nmid \mathfrak{p}} H^1(L_v^{un}, A)\right).$$

Now we define a representation attached to a Hecke character. Let ψ be a Hecke character of K with conductor f. Although our argument does not depend much on the infinity type, we let its infinity type be $(2k-1, 0)$.

Throughout this section we let $K(g)$ denote the ray class field of conductor g, and let $G_K = \mathrm{Gal}(\bar{\mathbb{Q}}/K)$.

Let p be a prime number. By [We] it is well-known that the field extension $K(\psi)$ of K obtained by adjoining the values of ψ is a finite extension, and if we fix an embedding $i_p : \bar{\mathbb{Q}} \to \mathbb{C}_p$, ψ extends continuously to a Galois character which factors through $\mathrm{Gal}(K(fp^\infty)/K)$. From now on, this Galois character will be denoted by the same letter ψ.

We fix a prime p that splits completely over $K/\mathbb{Q}$. Assume p is prime to f and $[K(f) : K]$. We let $\mathfrak{p}$ be the prime above p induced by the embedding i_p.

Let $d = [K(fp) : K]$. From now on, we let F be any extension of $\mathbb{Q}_p$ containing all the d-th roots of unity and the values of ψ under the embedding $i_p : \bar{\mathbb{Q}} \to \mathbb{C}_p$, and let O be O_F. We let $F(\psi)$ be the one-dimensional F-representation on which G_K acts through ψ (in other words, for $x \in F(\psi)$ and $\sigma \in G_K$, $\sigma \cdot x = \psi(\sigma|_{K(fp^\infty)})x$), and define V_ψ to be its $(-k+1)-$th Tate twist

$$V_\psi := F(\psi)(-k+1) = F(\psi)(\chi_{cyc}^{-k+1}),$$

and similarly define

$$T_\psi := O(\psi)(-k+1), \quad A_\psi := V_\psi/T_\psi = F/O(\psi)(-k+1). \tag{8.1}$$

Let ϕ be the Hecke character of K defined by

$$\phi(\mathfrak{a}) = \psi(\mathfrak{a})/N_{K/\mathbb{Q}}\mathfrak{a}^{k-1}$$

for every ideal $\mathfrak{a}$ prime to f. Its conductor is f and its infinity type is $(k, -k+1)$. Since $\psi(\mathfrak{a})\bar{\psi}(\mathfrak{a}) = N_{K/\mathbb{Q}}\mathfrak{a}^{2k-1}$, we have

$$V_\psi = F(\phi), \quad L(\phi^{-1}, 0) = L(\bar{\psi}, k).$$

Now, we will apply Iwasawa theory to relate the order of the Selmer group of A_ψ and the value of $L(\bar{\psi}, k)$. Every assumption we made is still in force. Let K_∞ be the maximal $\mathbb{Z}_p^2$-extension of K inside $K(fp^\infty)$. Because $[K(fp^\infty) : K_\infty]$ is prime to p, it is not hard to see that by the Hochschild-Serre spectral sequence we have

$$S(A_\psi/K_\infty) \cong S(A_\psi/K(fp^\infty))^{\mathrm{Gal}(K(fp^\infty)/K_\infty)}.$$

Let M be the maximal abelian extension of $K(fp^\infty)$ unramified outside $\mathfrak{p}$. Let $X = \mathrm{Gal}(M/K(fp^\infty))$. Since $G_{K(fp^\infty)}$ acts trivially on A_ψ, we have

$$S(A_\psi/K(fp^\infty)) \cong \mathrm{Hom}(X, A_\psi).$$

Thus

$$S(A_\psi/K_\infty) \cong \mathrm{Hom}(X(\phi^{-1})^\Delta, F/O)$$

for $\Delta = \mathrm{Gal}(K(fp^\infty)/K_\infty)$.

Proposition 8.3. *Let w_f be the group of the roots of unity of K congruent to 1 modulo f. If $w_f = \{1\}$, $\phi|_\Delta$ is not trivial. (Note that for a square-free $D > 4$, the only roots of unity in $K = \mathbb{Q}(\sqrt{-D})$ are ± 1.)*

Proof. First we note that if $w_f = \{1\}$, there is an elliptic curve E over $K(f)$ with complex multiplication by O_K such that its Hecke character $\psi_{E/K(f)}$ of $K(f)$ satisfies

$$\psi_{E/K(f)} = \varphi_E \circ N_{K(f)/K}$$

for a Hecke character φ_E of K of infinity type $(1,0)$ and conductor f (see [deS, p. 41]). Then we can write

$$\phi = \varphi_E^k \bar{\varphi}_E^{-k+1} \eta$$

for a finite character $\eta : I(f)/P_f \to \mathbb{C}^\times$. Through $I(f)/P_f \cong \mathrm{Gal}(K(f)/K)$, η induces a Galois character which we denote by the same letter. We also use the same letters φ_E and $\bar{\varphi}_E$ to denote the Galois characters of $\mathrm{Gal}(K(fp^\infty)/K)$ induced from them.

Since p splits completely over $K/\mathbb{Q}$, φ_E factors through $\mathrm{Gal}(K(f\mathfrak{p}^\infty)/K)$ and $\bar{\varphi}_E$ through $\mathrm{Gal}(K(f\bar{\mathfrak{p}}^\infty)/K)$, and $\varphi_E = \chi_{cyc}$ on the decomposition group $D_\mathfrak{p}$ of $\mathfrak{p}$ and $\bar{\varphi}_E = \chi_{cyc}$ on $D_{\bar{\mathfrak{p}}}$.

Since k or $k-1$ is not divisible by $p-1$, it follows that ϕ on $\mathrm{Gal}(K(fp)/K(f))$ is not trivial. □

From now on, we assume $w_f = 1$. We fix the elliptic curve E mentioned in the proof of Proposition 8.3. If we fix the Weierstrass model

$$y^2 = 4x^3 - g_2 x - g_3, \quad g_2, g_3 \in F$$

of E, we can find a lattice L of $\mathbb{C}$ with

$$g_2 = 60 \cdot \sum_{\omega \in L - \{0\}} \omega^{-4}, \quad g_3 = 140 \cdot \sum_{\omega \in L - \{0\}} \omega^{-6}.$$

Since all Galois conjugates of E are isogenous, by taking a Galois conjugate of E if necessary, we can assume $L = \Omega f$ for some $\Omega \in \mathbb{C}^\times$.

Let $U(fp^n)$ be the group of local units of $K(fp^n) \otimes_K K_\mathfrak{p}$ congruent to 1 modulo the primes above $\mathfrak{p}$ and U_∞ be $\varprojlim U(fp^n)$. Let $C(fp^n)$ be the closure of the group of elliptic units in $U(fp^n)$ and C_∞ be $\varprojlim C(fp^n)$. For the definition of elliptic units, see [deS, chapter 2, section 2].

Remark 8.4. We should note that when f is a power of a prime, the elements defined in [deS] are not units. However, we can easily fix this. See [Ru2, section 1].

For an algebra R, a finitely generated R-module is called pseudo-null if it is annihilated by an ideal of height 2. A pseudo-isomorphism of R-modules is a map with pseudo-null kernel and cokernel.

Let Λ be the Iwasawa algebra $O[[\mathrm{Gal}(K_\infty/K)]]$. It follows from the well-known classification theorem of Λ-modules that for every finitely generated torsion Λ-module Y we can find elements f_i of Λ and pseudo-isomorphisms

$$Y \to \oplus \Lambda/f_i\Lambda, \quad \oplus\Lambda/f_i\Lambda \to Y.$$

The characteristic ideal $\mathrm{char}_\Lambda(Y) := (\prod f_i)\Lambda$ is independent of the choice of the pseudo-isomorphism.

Theorem 8.5 (Rubin [Ru2], Theorem 4.1 (i))**.** *One has*

$$\mathrm{char}_\Lambda(X^{\phi|_\Delta}) = \mathrm{char}_\Lambda((U_\infty/C_\infty)^{\phi|_\Delta}).$$

The following is a generalization of [Ya]. Recall that Ω is the complex period of the elliptic curve E given by the Weierstrass model, and this number does not depend on anything but the conductor f and the Weierstrass model of E. In a sense, it does not depend much on the conductor either because we can use the same elliptic curve E for any Hecke character with conductor divisible by f.

Theorem 8.6 ([deS], chapter 2, Theorem 4.14)**.** *Let R be the ring of integers in $\mathbb{C}_p$ (i.e. p-adic numbers of non-negative p-adic valuation). There is a measure μ_f on $\mathrm{Gal}(K(fp^\infty)/K)$ and a p-adic period $\Omega_p \in R^\times$ such that for any Hecke character χ of conductor dividing fp^∞ and of type (k, j) with $0 \le -j < k$ we have*

$$\Omega_p^{j-k} \int_{\mathrm{Gal}(K(fp^\infty)/K)} \chi(\sigma) d\mu_f(\sigma) = \Omega^{j-k} (\frac{\sqrt{d_K}}{2\pi})^j G(\chi)(1 - \frac{\chi(\mathfrak{p})}{p}) L_{f\bar{\mathfrak{p}}}(\chi^{-1}, 0)$$

which is in $\bar{\mathbb{Q}}$.

Here, $L_{f\bar{\mathfrak{p}}}(\chi^{-1}, 0)$ is the sum over all ideals prime to $f\bar{\mathfrak{p}}$, and $G(\chi)$ is a certain Gauss sum which is 1 if the conductor of χ is prime to $\mathfrak{p}$.

For any integer n, a map

$$\mu_n : \mathrm{Gal}(K(fp^n)/K) \to O$$

given by

$$\mu_n(\sigma) = \mu_f(\sigma \cdot \mathrm{Gal}(K(fp^\infty)/K(fp^n)))$$

can be canonically identified with an element of the group ring $O[\mathrm{Gal}(K(fp^n)/K)]$ by $\sum \mu_n(\sigma)\sigma$. By taking the inverse limit, μ_f can be identified with an element g of

$$O[[\mathrm{Gal}(K(fp^\infty)/K)]].$$

Let g_ϕ be its image under

$$\phi|_\Delta : O[[\mathrm{Gal}(K(fp^\infty)/K)]] \to \Lambda.$$

We choose topological generators γ_1 of

$$\mathrm{Gal}(K(f\mathfrak{p}^\infty)/K(f\mathfrak{p}))$$

and γ_2 of

$$\mathrm{Gal}(K(f\bar{\mathfrak{p}}^\infty)/K(f\bar{\mathfrak{p}})).$$

Consider γ_1 and γ_2 as topological generators of $\mathrm{Gal}(K(fp^\infty)/K(fp))$ and identify Λ with the power series ring $O[[S,T]]$ by identifying $\gamma_1 = S+1$ and $\gamma_2 = T+1$. We let $g_\phi(S,T)$ be the power series corresponding to g_ϕ under the identification $\Lambda \cong O[[S,T]]$.

Proposition 8.7. *We have*

$$\mathrm{char}_\Lambda((U/C)^{\phi|_\Delta}) = (g_\phi).$$

Proof. Since $\phi|_\Delta$ is not trivial (Proposition 8.3), this follows from [deS, p. 105]. $\square$

Thus by Theorem 8.5 we have the following.

Proposition 8.8. *We have*

$$\mathrm{char}_\Lambda(X(\phi^{-1})^\Delta) = (g_\phi(\phi(\gamma_1)(S+1)-1, \phi(\gamma_2)(T+1)-1)).$$

Let K'_∞ be the maximal $\mathbb{Z}_p$-subextension of $K(f\mathfrak{p}^\infty)$. We can easily see that we can consider γ_1 as a topological generator of $\Gamma_1 := \mathrm{Gal}(K'_\infty/K)$. When Γ_i is the closed subgroup of $\mathrm{Gal}(K_\infty/K)$ generated by γ_i for $i = 1, 2$, we let Λ_i denote $O[[\Gamma_i]]$.

Proposition 8.9. *We have*

$$\mathrm{char}_{\Lambda_1}(X(\phi^{-1})^\Delta/(\gamma_2-1)X(\phi^{-1})^\Delta) = (g_\phi(\phi(\gamma_1)(S+1)-1, \phi(\gamma_2)-1)).$$

Proof. Let Y be a finitely generated torsion Λ-module with no non-trivial pseudo-null submodule. Let $f(S,T)$ denote a generator of $\mathrm{char}_\Lambda Y$.

By [Ru2, Lemma 6.2 (i)], the following are equivalent:

(1) $Y/(\gamma_2-1)Y$ is Λ_1-torsion,
(2) $f(S,0)$ is not 0 in Λ_1
(3) Y^{Γ_2} is a pseudo-null Λ-module.

If any of them is true, by [Ru2, Lemma 6.2 (ii)] we have

$$\mathrm{char}_{\Lambda_1}(Y/(\gamma_2-1)Y) = f(S,0)\,\mathrm{char}_{\Lambda_1}(Y^{\Gamma_2}).$$

However, Y^{Γ_2} is pseudo-null and by our assumption Y has no non-trivial pseudo-null submodule, thus $Y^{\Gamma_2}=0$.

By [Ru2, Theorem 5.3] (see also [P, chapter II]) X is $O[[\mathrm{Gal}(K(fp^\infty)/K)]]$-torsion and has no non-trivial pseudo-null submodule. Of course the same holds when we twist X by ϕ^{-1}. Thus our claim follows. □

Let $M^\vee$ denote the Pontryagin dual $\mathrm{Hom}_O(M, F/O)$.

Proposition 8.10. *If $k-1$ is not divisible by $p-1$, then $S(A_\psi/K'_\infty)^\vee$ has no non-trivial finite Λ_1-submodule and*

$$\mathrm{char}_{\Lambda_1} S(A_\psi/K'_\infty)^\vee = (g_\phi(\phi(\gamma_1)(S+1)-1, \phi(\gamma_2)-1)).$$

Proof. First we consider the following commutative diagram.

$$\begin{array}{ccccc}
0\to \quad S(A_\psi/K'_\infty) & \to & H^1(K'_\infty, A_\psi) & \to & \prod\limits_{w'\nmid \mathfrak{p}} H^1(K'^{,un}_{\infty,w'}, A_\psi) \\
\downarrow & & \downarrow & & \downarrow \\
0\to \quad S(A_\psi/K_\infty)^{\Gamma_2} & \to & H^1(K_\infty, A_\psi)^{\Gamma_2} & \to & \prod\limits_{w\nmid \mathfrak{p}} H^1(K^{un}_{\infty,w}, A_\psi).
\end{array}$$

As we have seen, by the Hochschild-Serre spectral sequence the middle vertical arrow is an isomorphism. Since any prime of K not above p is unramified over K_∞/K and finitely decomposed over K'_∞/K, the right vertical arrow is injective for primes not above p.

For a prime of K_∞ above $\bar{\mathfrak{p}}$ (which we also denote by $\bar{\mathfrak{p}}$ since there is only one), since we assume $k-1$ is not divisible by $p-1$ and ϕ is equal to $\varphi_E^k\bar{\varphi}_E^{-k+1}\eta$, $G_{K^{un}_{\infty,\bar{\mathfrak{p}}}}$ does not act trivially on any non-trivial subgroup of A_ψ. Thus by the Hochschild-Serre spectral sequence the right vertical arrow is injective for $\bar{\mathfrak{p}}$. Hence by the Snake Lemma the left vertical arrow is an isomorphism.

Thus

$$S(A_\psi/K'_\infty)^\vee \cong X(\phi^{-1})^\Delta/(\gamma_2-1)X(\phi^{-1})^\Delta$$

and we know its characteristic ideal from Proposition 8.9.

The proof of [Ru2, Lemma 11.15] shows that $X(\phi^{-1})/(\gamma_2-1)X(\phi^{-1})$ has no non-trivial finite Λ_1-submodule if $X/(\gamma_2-1)X$ has no non-trivial finite Λ_1-submodule. We have $X/(\gamma_2-1)X \cong \mathrm{Gal}(M(f\mathfrak{p}^\infty\bar{\mathfrak{p}})/K(f\mathfrak{p}^\infty\bar{\mathfrak{p}}))$ where $M(f\mathfrak{p}^\infty\bar{\mathfrak{p}})$ is the maximal abelian extension of $K(f\mathfrak{p}^\infty\bar{\mathfrak{p}})$ unramified outside $\mathfrak{p}$. By [Ru2, Theorem 5.3 (v)] our claim follows. □

Remark 8.11. We can obtain the characteristic ideal of $S(A_\psi/K'_\infty)^\vee$ without the assumption $k-1$ is not divisible by $p-1$, thus without this assumption we can prove that $L_{f\bar{\mathfrak{p}}}(\phi^{-1},0)\neq 0$ if and only if $S(A_\psi/K)$ is finite (which is equivalent to $\mathrm{Sel}_p(A_\psi/K)$ being finite, as we will see later).

We need to examine the Bloch-Kato local conditions more closely. For a prime $v \nmid p$ it is clear that $H^1_f(K_v, A_\psi) = 0$.

Recall that we can write $\phi = \phi_E^k \bar{\phi}_E^{-k+1}\eta$ where $\phi_E = \chi_{cyc}$ on $D_\mathfrak{p}$ and $\bar{\phi}_E = \chi_{cyc}$ on $D_{\bar{\mathfrak{p}}}$. (See the proof of Proposition 8.3.) By [BK, section 3] we have

$$\dim_F H^1_f(K_v, V) = \dim_F (V \otimes B_{DR})^{G_{K_v}} / (V \otimes B^0_{DR})^{G_{K_v}}$$

for $v|p$, thus we can see $\dim_F H^1_f(K_\mathfrak{p}, V_\psi) = 1$ and $\dim_F H^1_f(K_{\bar{\mathfrak{p}}}, V_\psi) = 0$.

On the other hand, by the local Euler characteristic formula and Tate local duality we have

$$\dim_F H^1(K_v, V) = [K_v : \mathbb{Q}_p] \dim_F V + \dim_F H^0(K_v, V) + \dim_F H^0(K_v, V^*)$$

where V^* denotes $\mathrm{Hom}(V, F(1))$. It is clear that $V_\psi^{G_{K_\mathfrak{p}}} = 0$ and $(V_\psi^*)^{G_{K_\mathfrak{p}}} = 0$, thus we have $\dim_F H^1(K_\mathfrak{p}, V_\psi) = 1$.

Thus we can conclude that $H^1_f(K_{\bar{\mathfrak{p}}}, A_\psi) = 0$ and $H^1_f(K_\mathfrak{p}, A_\psi)$ is the image of

$$H^1(K_\mathfrak{p}, V_\psi) \to H^1(K_\mathfrak{p}, A_\psi).$$

By the long exact sequence of cohomology groups induced from $T_\psi \to V_\psi \to A_\psi$, we find that the cokernel of the map above is $H^2(K_\mathfrak{p}, T_\psi)_{tors}$. We have

$$\left|H^2(K_\mathfrak{p}, T_\psi)\right| = \left|H^0(K_\mathfrak{p}, \mathrm{Hom}_O(T_\psi, F/O(1)))\right| = \left|F/O(\phi_E^{-k}\bar{\phi}_E^{k-1}\eta^{-1})(1)^{G_{K_\mathfrak{p}}}\right|$$

by local Tate duality, and if we assume $p-1 \nmid k-1$, this last group is trivial, thus we can see $H^1_f(K_\mathfrak{p}, A_\psi) = H^1(K_\mathfrak{p}, A_\psi)$ if $p-1 \nmid k-1$.

Also, we need to examine the local conditions of $S(A_\psi/K'_\infty)$.

Every prime $v \neq \mathfrak{p}$ of K is unramified and finitely decomposed over K'_∞/K, thus for any $w|v$, $H^1(K'^{un}_{\infty,w}/K'_{\infty,w}, A_\psi^{G_{K'^{un}_{\infty,w}}}) = 0$.

Our discussion so far helps find the kernel and cokernel of

$$\mathrm{Sel}_p(A_\psi/K) \to S(A_\psi/K'_\infty)^{\mathrm{Gal}(K'_\infty/K)},$$

however for a precise result, we need more techniques. First, we need a generalized version of the Cassels-Tate theorem. Let Σ be a set of places of K including ∞ and the prime divisors of pf. We let K_Σ denote the maximal extension of K unramified outside Σ. We define $T_\psi^* := \mathrm{Hom}_O(A_\psi, F/O(1))$, $V_\psi^* := T_\psi^* \otimes \mathbb{Q}_p$, and $A_\psi^* := V_\psi^*/T_\psi^*$.

We note that for any prime v there is the non-degenerate local Tate pairing

$$H^1(K_v, A_\psi) \times H^1(K_v, T_\psi^*) \to F/O.$$

We define $H^1_f(K_v, T_\psi^*)$ as the exact annihilator of $H^1_f(K_v, A_\psi)$ with respect to this pairing. One consequence is that $H^1_f(K_v, T_\psi^*)$ contains $H^1(K_v, T_\psi^*)_{tors}$. Another consequence is the following: Define

$$S(T_\psi^*/K) := \ker(H^1(K_\Sigma/K, T_\psi^*) \to \prod_{v \in \Sigma} H^1(K_v, T_\psi^*)/H^1_f(K_v, T_\psi^*)).$$

Note that $V_\psi^* \cong F(\psi^{-1})(\chi_{cyc}^k) \cong F(\bar{\psi})(-k+1)$. It is not hard to see $S(A_\psi/K)$ is finite if and only if $S(T_\psi^*/K)$ is finite.

Proposition 8.12. *If $S(T_\psi^*/K)$ is finite,*

$$H^1(K_\Sigma/K, A_\psi) \to \prod_{v \in \Sigma} H^1(K_v, A_\psi)/H_f^1(K_v, A_\psi)$$

is surjective.

Proof. Similar to [Gre, Proposition 4.13] and [K, Proposition 13]. □

For the following proposition, recall that $H^1(K'^{un}_{\infty,w}/K'_{\infty,w}, A_\psi^{G_{K'^{un}_{\infty,w}}}) = 0$ for any prime w not above $\mathfrak{p}$.

Proposition 8.13. *If $k-1$ is not divisible by $p-1$, we have*

$$\#\operatorname{Sel}_p(A_\psi/K) \prod_{v \in \Sigma, v \nmid p} \# H^0(K_v, A_\psi) = \# S(A_\psi/K'_\infty)^{\Gamma_1}.$$

Proof. Recall our discussion about the local conditions. Consider the following commutative diagram.

$$\begin{array}{ccccccc}
0 \to & \operatorname{Sel}_p(A_\psi/K) & \to & H^1(K_\Sigma/K, A_\psi) & \to & \prod_{v \in \Sigma, v \neq \mathfrak{p}} H^1(K_v, A_\psi) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \to & S(A_\psi/K'_\infty)^{\Gamma_1} & \to & H^1(K_\Sigma/K'_\infty, A_\psi)^{\Gamma_1} & \to & \prod_{w | \Sigma, w \nmid \mathfrak{p}} H^1(K'_{\infty,w}, A_\psi).
\end{array}$$

Since $A^{G_{K'_\infty}} = 0$, by Hochschild-Serre, the middle vertical map is an isomorphism.

As for the right vertical map, first we note $\operatorname{Gal}(K'_{\infty,w}/K_v) \cong \mathbb{Z}_p$ for every w. Let γ_w be its topological generator. For $H = A_\psi^{G_{K'_{\infty,w}}}$ we have a short exact sequence

$$0 \to H^{G_{K_v}} \to H \xrightarrow{\gamma_w - 1} H \to H/(\gamma_w - 1)H \to 0.$$

Since H is a finite group, $\# H/(\gamma_w - 1)H = \# H^{G_{K_v}}$. By [Ru3, Lemma B.2.8],

$$H^1(K'_{\infty,w}/K_v, H) \cong H/(\gamma_w - 1)H.$$

Note that by Proposition 8.12 the last map in the top of the commutative diagram above is surjective. Thus by the Snake Lemma, our claim follows. □

For a prime v of K, let a_v denote $\# H^0(K_v, A_\psi)$ which plays the role of the local Tamagawa number.

We let v_p denote the normalized p-adic valuation. Recall the definition of the complex period Ω, which depends only on the conductor f. Also, recall that ψ is a Hecke character of infinity type $(2k-1, 0)$ and conductor f, and recall the definition of A_ψ.

Theorem 8.14. *Recall we assume $w_f = \{1\}$ (see Proposition 8.3). Let p be a prime that splits completely over $K/\mathbb{Q}$ and is prime to f.*

(1) $\operatorname{Sel}_p(A_\psi/K)$ *is finite if and only if* $L_{f\bar{\mathfrak{p}}}(\bar{\psi}, k) \neq 0$.

(2) *Additionally, assume $k-1$ and k are not divisible by $p-1$. If $L_{f\bar{\mathfrak{p}}}(\bar{\psi}, k) \neq 0$, then*

$$v_p(\#\,\mathrm{Sel}_p(A_\psi/K)\prod_{v\in\Sigma} a_v) = [O:\mathbb{Z}_p]\cdot v_p\left((\frac{\sqrt{d_K}}{2\pi})^{-k+1}(1-\frac{\psi(\mathfrak{p})}{p^k})\frac{L_{f\bar{\mathfrak{p}}}(\bar{\psi},k)}{\Omega^{2k-1}}\right).$$

Proof. The first claim follows from Proposition 8.10 (also see Remark 3).

Let B be a Λ_1-torsion module with no non-trivial finite Λ_1-submodule and let $f(S) \in \Lambda_1 \cong O[[S]]$ be a generator of the characteristic ideal of B. It is an exercise in Iwasawa theory to show $v_p(\#B/(\gamma_1 - 1)B) = v_p(f(0))$. Note that since $k-1$ and k are not divisible by $p-1$, $H^0(K_v, A_\psi) = 0$ for $v = \mathfrak{p}, \bar{\mathfrak{p}}$. The rest follows from Theorem 8.6 and Propositions 8.10 and 8.13. □

Remark 8.15. i) The presence of $[O : \mathbb{Z}_p]$ on the right side in (2) is due to the fact that the groups we deal with are actually O-modules.

ii) It is likely that (2) should be true for almost all p, but one needs additional techniques for other primes.

9. Complex periods

To discuss the issue of periods, first we need to explain the Tamagawa number defined by Bloch and Kato [BK] because in their paper a complex period is simply the Tamagawa number for infinite places. And, to that end, we need to state their conjecture in its original form (rather than a more common version like Theorem 8.14). For the readers not familiar with the Bloch-Kato conjecture, one can consider it as an analogue of the Birch and Swinnerton-Dyer conjecture for motives.

First, we need to state the conjecture precisely.

Let (V, D) be a motivic pair of $\mathbb{Q}$-vector spaces (see [BK, Definition 5.5]). For $p \leq \infty$, let V_p denote $V \otimes \mathbb{Q}_p$ and D_p denote $D \otimes \mathbb{Q}_p$. Let $\mathbb{A} = \mathbb{A}(\mathbb{Q})$. Here D has a filtration structure which induces the (Hodge) filtration of $DR(V \otimes \mathbb{Q}_p)$ for $p \leq \infty$. To state the Bloch-Kato conjecture we need to choose a lattice M of V such that $M \otimes \hat{\mathbb{Z}}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-stable in $V \otimes \mathbb{A}_f$ (see [BK, p. 372]). We define the *rational points* by

$$A(K_v) := H^1_f(K_v, M \otimes \hat{\mathbb{Z}}) \quad \text{for a finite place } v,$$
$$A(\mathbb{C}) := ((D_\infty \otimes_{\mathbb{R}} \mathbb{C})/((D^0_\infty \otimes_{\mathbb{R}} \mathbb{C}) + M))^+$$

where $D_\infty = D \otimes \mathbb{R}$ and the inclusion $M \to D_\infty \otimes_{\mathbb{R}} \mathbb{C}$ is given by the identification $D_\infty \otimes_{\mathbb{R}} \mathbb{C} = V_\infty \otimes \mathbb{C}$. (For the definition of H^1_f, see [BK].)

Note that for any $p \leq \infty$ the exponential map

$$\exp : D_p/D^0_p \otimes K_v \to A(K_v) \quad (v|p)$$

is a local isomorphism.

Fix an isomorphism $\omega : \det(D/D^0) \cong \mathbb{Q}$. This isomorphism induces a Haar measure on D_p/D^0_p, thus by the aforementioned exp it induces a Haar measure $\mu_{v,\omega}$ on $A(K_v)$ for every (finite or infinite) place v. Let S be a sufficiently large set of a finite number of places of K including ∞ and the ramified primes. Let l be a fixed prime number such that no prime in S is lying above it, and for any prime v of K let $P_v(V_l, X) = \det(1 - \mathrm{Frob}_v^{-1} X | V_l^{I_v})$ and let $L_S(V, s) = \prod_{v \notin S} P_v(V_l, Nv^{-s})^{-1}$. Then by [BK, Theorem 4.1] we have

$$L_S(V, 0)^{-1} = \prod_{p \notin S} \mu_{v,\omega}(A(K_v)). \tag{9.1}$$

if $L_S(V, s)$ is convergent at $s = 0$.

We will not discuss the (somewhat conjectural) group of global rational points $A(K)$ in detail, but we only mention that $A(K)_{tor} \cong H^0(K, M \otimes \mathbb{Q}/\mathbb{Z})$.

We define

$$\mathrm{Tam}(M) = \mu((\prod A(K_v)/A(K)). \tag{9.2}$$

The following is the Bloch-Kato conjecture, which should be distinguished from the other Bloch-Kato conjecture in K-theory.

Conjecture 9.1. (1) $\operatorname{rank} A(K) = \operatorname{ord}_{s=0} L(V, s)$.

(2) $\mathrm{Tam}(M) = \dfrac{\#(H^0(K, M^* \otimes \mathbb{Q}/\mathbb{Z}(1)))}{\#Ш(M/K)}$.

But perhaps most readers are more familiar with the following:

Conjecture 9.2. *Let p be any prime number. Define*

$$\operatorname{corank}_{\mathbb{Z}_p} A = \operatorname{rank}_{\mathbb{Z}_p} \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$$

for any abelian group A. Then

(1) $\operatorname{corank}_{\mathbb{Z}_p} \mathrm{Sel}_p(M/K) = \operatorname{ord}_{s=0} L(V, s)$.

(2)

$$L_S(V, 0) = \frac{\# \mathrm{Sel}_p(M/K)}{\#H^0(K, M^* \otimes \mathbb{Q}/\mathbb{Z}(1))} \mu_{\infty,\omega}(A(\mathbb{C})/A(K)) \prod_{v \in S - \infty} \mu_{v,\omega}(A(K_v)).$$

Here (2) is induced by (9.1). Note that the right side does not depend on the choice of ω.

For a finite place v of K let a_v denote $\#H^0(K_v, M \otimes \mathbb{Q}/\mathbb{Z})$. By [BK, Lemma 5.10] and its discussion on p. 373, $\mu_{v,\omega}(A(K_v))$ is the product of some l-power and $H^0(K_v, M \otimes \prod_{h \neq l} \mathbb{Q}_h/\mathbb{Z}_h)$ where v lies above l. Thus for any prime $p \neq l$, we have

$$\mu_{v,\omega}(A(K_v)) = \#H^0(K_v, M \otimes \mathbb{Q}_p/\mathbb{Z}_p) \tag{9.3}$$

up to a p-adic unit.

Now we will construct the motivic pair attached to our Hecke character ψ. This is constructed by Schappacher [Sc] following a much more general result of Jannsen [J] (see also Scholl [S]), but we need a more specific construction.

Now we define a motivic pair (V, D) and a lattice $M \subset V$ attached to ψ. We let

$$V_p = \mathbb{Q}_p(\phi_{\mathfrak{p}}) \oplus \mathbb{Q}_p(\phi_{\bar{\mathfrak{p}}})$$

if $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ in K, and

$$V_p = K_p(\phi_p)$$

if p is inert over $K/\mathbb{Q}$. (This is a little misleading because the values of $\phi_{\mathfrak{p}}$ and so on are not necessarily in $\mathbb{Q}_p$ or K_p, but it seems better to keep the notation simple.)

We define V_{DR} and V_B as follows. Recall the elliptic curve E/F we chose earlier. Also recall that by the Weierstrass model we can identify E with $\mathbb{C}/L$ where $L = \Omega f$ for some $\Omega \in \mathbb{C}^\times$ and some ideal $f \subset O_K$. Let $F \in f$ be some non-zero number prime to p. Let $(M_B(E), M_{DR}(E), M_p)$ be the motive of E. Then $M_B(E)$ is given by

$$M_B(E) = H_B^{-1}(E) = \mathbb{Q}X_1 \oplus \mathbb{Q}X_2$$

where X_1 and X_2 are the cycles obtained by the directed line segments from 0 to ΩF and from 0 to $\sqrt{-D}\Omega F$ on $\mathbb{C}/L$.

The de Rham realization $M_{DR}(E)$ is given by

$$M_{DR}(E) = \mathbb{Q}\hat{\omega}_1 \oplus \mathbb{Q}\hat{\omega}_2$$

where $\hat{\omega}_1$ corresponds to $\hat{dz}$ and $\hat{\omega}_2$ corresponds to $\hat{d\bar{z}}$ such that

$$Fil^i M_{DR} = 0, \quad i > 0,$$
$$Fil^0 M_{DR} = \mathbb{Q}\omega_2,$$
$$Fil^i M_{DR} = M_{DR}, \quad \text{otherwise.}$$

Similarly, let $(M_B(G_m), M_{DR}(G_m), M_p(G_m)$ be the motive of the multiplicative group G_m. First, identify $\exp(\mathbb{C})$ with $\mathbb{C}/2\pi i\mathbb{Z}$, then $M_B(G_m)$ is

$$M_B(G_m) = \mathbb{Q}X$$

where X is the directed path from 0 to $2\pi i$ on $\mathbb{C}/2\pi i\mathbb{Z}$. The de Rham realization $M_{DR}(G_m)$ is given by $\mathbb{Q}\hat{\epsilon}$ where $\hat{\epsilon}$ is the dual basis of the differential $\epsilon = dz/z$.

Now similar to [Gu2, section 2], for any subset π of $\{1, 2, \ldots, n\}$ we define

$$\pi(i) = 1 \quad \text{if } i \in \pi,$$
$$\pi(i) = 2 \quad \text{if } i \notin \pi.$$

As in [Gu2] we define the Betti realization

$$V_B := (\mathbb{Q}e_1 \oplus \mathbb{Q}e_2) \otimes \mathbb{Q}X^{1-k}$$

where e_1 and e_2 are defined by

$$\frac{1}{(-D)^{2k-1}} \sum_{\pi \subset \{1,\ldots,2k-1\}} D^{\#\pi/2} X_{\pi(1)} \otimes \cdots \otimes X_{\pi(2k-1)} = e_1 + \frac{1}{\sqrt{-D}} e_2.$$

And, define the de Rham realization

$$V_{DR} = (\mathbb{Q}\hat{\omega}_1^{\otimes 2k-1} \oplus \mathbb{Q}\hat{\omega}_2^{\otimes 2k-1}) \otimes \mathbb{Q}\hat{\epsilon}^{\otimes -k+1}.$$

Now we define a strongly divisible (see [BK, p. 362] for its definition) lattice M inside V by

$$M_B = (\mathbb{Z}e_1 + \mathbb{Z}e_2) \otimes \mathbb{Z}X^{\otimes -k+1}$$

such that $M_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is mapped onto the lattice M_p of V_p

$$\mathbb{Z}_p(\phi_{\mathfrak{p}}) \oplus \mathbb{Z}_p(\phi_{\bar{\mathfrak{p}}})$$

if p splits and

$$O_{K,p}(\phi_p)$$

otherwise.

On the other hand, define a vector space with a filtration D by $D = \mathbb{Q}x_1 \oplus \mathbb{Q}x_2$ such that

$$Fil^i(D) = 0, \quad i > k-1,$$

$$Fil^i(D) = \mathbb{Q}x_2, \quad -k < i \leq k-1,$$

$$Fil^i(D) = D, \quad i \leq -k.$$

Similar to [Gu2, Lemma 3.1] we obtain

$$\mu_{\infty,\omega}(A(\mathbb{C})/A(K)) = \alpha(\sqrt{d_K}/2\pi)^{2(-k+1)}(N_{K/\mathbb{Q}}F\Omega\bar{\Omega})^{2k-1}$$

for some $\alpha \in \mathbb{Q}^\times$. (In [Gu2], $\mu_{\infty,\omega}$ is normalized. This process seems equivalent to multiplying $\mu_{\infty,\omega}$ by some $\alpha \in \mathbb{Q}^\times$. This makes sense because $\mu_{\infty,\omega}(A(\mathbb{C})/A(K))$ plays the role of a complex period, and a complex period is in general defined up to some algebraic number.) Assume p is prime to αF. If $L_{f\bar{\mathfrak{p}}}(V,0)$ is nonvanishing, Conjecture 9.2 can be rewritten as follows.

Conjecture 9.3. (1) $\mathrm{Sel}_p(M_p \otimes \mathbb{Q}/\mathbb{Z}/K)$ *is finite.*

(2) $L_{f\bar{\mathfrak{p}}}(V,0) = \dfrac{\#\,\mathrm{Sel}_p(M_p \otimes \mathbb{Q}/\mathbb{Z}/K)}{\#H^0(K, M^* \otimes \mathbb{Q}_p/\mathbb{Z}_p(1))}(\sqrt{d_K}/2\pi)^{2(-k+1)}(\Omega\bar{\Omega})^{2k-1}\prod_v a_v$ *up to a p-adic unit.*

By the functional equation we have $L(\phi^{-1},0) = L(\bar{\psi},k)$, and we also note that $L_S(V,0) = L(\phi^{-1},0) \cdot L(\bar{\phi}^{-1},0)$. Similarly, we note that $\mathrm{Sel}_p(M/K) \cong \mathrm{Sel}_p(A_\psi) \oplus \mathrm{Sel}_p(A_{\bar{\psi}})$. Since $L(\phi^{-1},0) = \overline{L(\bar{\phi}^{-1},0)}$ and $\mathrm{Sel}_p(A_\psi) \cong \mathrm{Sel}_p(A_{\bar{\psi}})$, we see that Conjecture 9.3 follows from Theorem 8.14 under the assumed conditions.

10. Proofs of Theorem 1.7 and Corollary 1.8

Proof of Theorem 1.7. Liu and Xu [LX] (extending earlier work of Miller and Yang [MiY]) proved that if $(2k-1, h(-D)) = 1$ and $|d| \ll_{k,\epsilon} D^{\frac{1}{12}-\epsilon}$, then

$$\#\{\psi_{d,k} \in \Psi_{d,k} : \; L(\psi_{d,k}, k) \neq 0\} = h(-D). \tag{10.1}$$

Part (1) follows by combining Theorem 1.5 with (10.1). Similarly, part (2) follows by combining Theorem 1.5 with Theorem 1.4. $\square$

Proof of Corollary 1.8. Recall the $\mathfrak{p}^\infty$-descent sequence

$$0 \to A(D)(H) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{\mathfrak{p}}(A(D)/H) \to Ш(A(D)/H)_{\mathfrak{p}} \to 0.$$

One can show that $\mathrm{Sel}_{\mathfrak{p}}(A(D)/H) \cong \mathrm{Sel}_p(\chi_H/H)$, and by Shapiro's lemma

$$\mathrm{Sel}_p(\chi_H/H) \sim \prod_{\xi \in \mathrm{CL}(K)^\wedge} \mathrm{Sel}_p(\psi_1\xi/K) = \prod_{\psi_1 \in \Psi_{1,1}} \mathrm{Sel}_p(\psi_1/K)$$

where $\sim$ means there is a homomorphism with finite kernel and cokernel. Thus Theorem 1.7 (1) implies that $\mathrm{Sel}_{\mathfrak{p}}(A(D)/H)$ is finite, which implies that $A(D)(H)$ and $Ш(A(D)/H)_{\mathfrak{p}}$ are finite. $\square$

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School of Mathematics, Victoria University of Wellington, P.O. Box 600, Wellington 6140 New Zealand
E-mail address: byoung.du.kim@mcs.vuw.ac.nz

Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706
E-mail address: masri@math.wisc.edu
E-mail address: thyang@math.wisc.edu