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Special values of Green functions at big CM points

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We give a formula for the values of automorphic Green functions on the special rational 0-cycles (big CM points) attached to certain maximal tori in the Shimura varieties associated to rational quadratic spaces of signature $(2d, 2)$. Our approach depends on the fact that the Green functions in question are constructed as regularized theta lifts of harmonic weak Maass forms, and it involves the Siegel-Weil formula and the central derivatives of incoherent Eisenstein series for totally real fields. In the case of a weakly holomorphic form, the formula is an explicit combination of quantities obtained from the Fourier coefficients of the central derivative of the incoherent Eisenstein series. In the case of a general harmonic weak Maass form, there is an additional term given by the central derivative of a Rankin-Selberg type convolution.

1 Introduction

In 1985, Gross and Zagier discovered a beautiful factorization formula for singular moduli [16]. This has inspired a lot of interesting work, including Dorman’s generalization to odd discriminants [13], Elkies’s examples on Shimura curves [14] and Lauter’s conjecture on the Igusa j-invariants ([15], [40], [39]), among others. In his thesis, Schofer [37] proved a much more general factorization formula for the ‘small’ CM values of Borcherds modular functions on a Shimura variety of orthogonal type via regularized theta liftings. The proof is very natural and is based on a method introduced in [20]. Two of the authors adapted the same idea to study the ‘small’ CM values of automorphic Green functions and discovered a direct link between the CM value and the central derivative of a certain Rankin-Selberg L-function. This direct link is used to give a different proof of the well-known Gross-Zagier formula [10]. Here ‘small’ means that the CM cycles are associated to imaginary

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quadratic fields. On the other hand, the two authors also extended Gross and Zagier's factorization formula, using a method close to Gross and Zagier's original idea, to 'big' CM values of some Hilbert modular functions on a Hilbert modular surface. Here 'big' means that the CM cycle is associated to a maximal torus of the reductive group giving the Hilbert modular surface.

A motivating question for this paper is whether this 'big' CM value result can also be derived using the regularized theta lifting method in [37] and [10], which is more natural and simpler. While the small CM cycles are constructed systematically and associated to rational negative two planes in the quadratic space defining the Shimura variety, no big CM cycles are constructed this way. In Section 2, we describe a way to construct big CM cycles in some special Shimura varieties (including Hilbert modular surfaces), and study their Galois conjugates. Such CM cycles are associated to CM fields of degree $2d+2$. In Sections 3–5, we extend the CM value result in [10] to this situation. In Section 6, we restrict to the special case of Hilbert modular surfaces and give a new proof of the main results in [9] and a generalization. Actually, to get the CM cycles in [9] from our present construction is not straightforward and quite interesting. An arithmetic application is given at the end of Section 6. We now describe this work in more detail.

Let (V, Q_V) be a rational quadratic space of signature $(2d, 2)$ for some positive integer $d \ge 1$. Let $G = \mathrm{GSpin}(V)$ and let $K \subset G(\hat{\mathbb{Q}})$ be a compact open subgroup*. Let $\mathbb{D}$ be the associated Hermitian domain of oriented negative 2-planes in $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$, and let

$$X_K = G(\mathbb{Q}) \backslash \big(\mathbb{D} \times G(\hat{\mathbb{Q}})/K \big) \tag{1.1}$$

be the complex points of the associated Shimura variety, which has a canonical model over $\mathbb{Q}$. Assume that there is a totally really number field F of degree $d+1$ and a two-dimensional F-quadratic space (W, Q_W) of signature

$$\mathrm{sig}(W) = ((0,2), (2,0), \cdots, (2,0))$$

with respect to the $d+1$ embeddings $\{\sigma_j\}_{j=0}^{d}$ such that

$$V = \mathrm{Res}_{F/\mathbb{Q}} W, \quad Q_V(x) = \mathrm{tr}_{F/\mathbb{Q}} Q_W(x).$$

Then there is an orthogonal direct sum decomposition

$$V(\mathbb{R}) = \bigoplus_j W_{\sigma_j}, \quad W_{\sigma_j} = W \otimes_{F,\sigma_j} \mathbb{R}.$$

The negative 2-plane W_{σ_0} gives rise to two points $z_0^{\pm}$ in $\mathbb{D}$. Let T be the preimage of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(W) \subset \mathrm{SO}(V)$ in G. Then T is a maximal torus associated to the CM number field $E = F(\sqrt{-\det W})$, and we obtain a 'big'

*We write $\hat{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ for the finite adèles of $\mathbb{Q}$, where $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$.

CM cycle in X_K:

$$Z(W, z_0^{\pm}) = T(\mathbb{Q})\backslash\big(\{z_0^{\pm}\} \times T(\hat{\mathbb{Q}})/K_T\big),$$

where $K_T = T(\hat{\mathbb{Q}}) \cap K$. The CM cycle $Z(W, z_0^{\pm})$ is defined over F, and the formal sum $Z(W)$ of all its Galois conjugates is a 0-cycle in X_K is defined over $\mathbb{Q}$. We refer to Section 2 for details.

Let L be an even integral lattice in V such that K preserves L and acts trivially on L'/L, where L' is the dual lattice. Let S_L be the space of locally constant functions on $\hat{V} = V \otimes \hat{\mathbb{Q}}$ which are $\hat{L}$-invariant and have support in $\hat{L}'$, and let ρ_L be the associated 'Weil representation' of $\mathrm{SL}_2(\mathbb{Z})$ on it. For each harmonic weak Maass form $f \in H_{1-d,\bar{\rho}_L}$, so that f is valued in S_L, there is a corresponding special divisor $Z(f)$ (determined by the principal part of f) and an automorphic Green function $\Phi(f)$ which is constructed in [6] as a regularized theta lift of f (see Section 3). On the other hand, associated to L, there is also an incoherent ($S_L^\vee$-valued and normalized) Hilbert Eisenstein series $E^*(\vec{\tau}, s, L, \mathbf{1})$ of parallel weight 1 (see Section 4) for the field F. Its diagonal restriction to $\mathbb{Q}$ is a weight $d+1$ non-holomorphic modular form with representation ρ_L. Due to the incoherence, $E^*(\vec{\tau}, 0, L, \mathbf{1}) = 0$. Let $\mathcal{E}(\tau, L)$ be the 'holomorphic part' of $E^{*,\prime}(\tau^\Delta, 0, L, \mathbf{1})$, where, for $\tau \in \mathbb{H}$, we put $\tau^\Delta = (\tau, \cdots, \tau) \in \mathbb{H}^{d+1}$. Finally define the generalized Rankin-Selberg L-function

$$\mathcal{L}(s, \xi(f), L) = \langle E^*(\tau^\Delta, s, L, \mathbf{1}), \xi(f)\rangle_{\mathrm{Pet}} \tag{1.2}$$

to be the Petersson inner product of the pullback of the Eisenstein series and the holomorphic cusp form $\xi(f)$ of weight $d+1$, given by the differential operator $\xi(f) = 2iv^{1-d}\overline{\frac{\partial f}{\partial \bar{\tau}}}$. In Section 5, we prove the following general formula.

Theorem 1.1. Let the notation be as above. Then

$$\Phi(Z(W), f) = \frac{\deg Z(W, z_0^{\pm})}{\Lambda(0, \chi_{E/F})} \left(\mathrm{CT}[\langle f^+(\tau), \mathcal{E}(\tau, L)\rangle] - \mathcal{L}'(0, \xi(f), L)\right). \tag{1.3}$$

Here $\chi_{E/F}$ is the quadratic Hecke character of F associated to E/F, f^+ is the holomorphic part of f, and $\mathrm{CT}[\langle f^+(\tau), \mathcal{E}(\tau, L)\rangle]$ is the constant term of

$$\langle f^+(\tau), \mathcal{E}(\tau, L)\rangle = \sum_{\mu \in L'/L} f^+(\tau, \mu)\, \mathcal{E}(\tau, L, \mu),$$

where $f^+(\tau, \mu)$ is the μ-component of f^+, and $\mathcal{E}(\tau, L, \mu)$ is the μ-component of $\mathcal{E}(\tau, L)$. □

In the special case that f is weakly holomorphic, i.e., when $\xi(f) = 0$, $\Phi(f)$ is the Petersson norm of a meromorphic modular form $\Psi(f)$ on X_K given by the Borcherds lift of f. In this case, the second summand on the right hand side of (1.3) vanishes and the first summand gives an explicit formula for the evaluation of $\Psi(f)$ on the CM cycle $Z(W)$. An important point here is that this result shows that the Fourier coefficients of the modular forms of parallel weight 1 arising from the central derivatives of incoherent Eisenstein series for arbitrary totally real fields F carry interesting arithmetic information – about the factorization of $\Psi(Z(W), f)$ in the present case, and ultimately about arithmetic intersection numbers.

Note that, in general, the first summand $\mathrm{CT}[\langle f^+(\tau), \mathcal{E}(\tau, L)\rangle]$ is of arithmetic nature and this theorem suggests two interesting conjectures about arithmetic intersection numbers and Faltings heights of big CM cycles, see Conjectures 5.4 and 5.5. In fact, our result can be viewed as a part of the calculation of the height pairing between the rational 0-cycles defined by big CM points and a certain linear combination of special divisors, or more precisely, a class in the first arithmetic Chow group, determined by f.

Also note that, in contrast with the situation in [10], the function $\mathcal{L}(s, \xi(f), L)$ is not a standard Rankin-Selberg integral, since it involves the pullback of a Hilbert modular Eisenstein series. We expect that it is related to a Langlands L-function for the group G and hope to pursue this idea in a subsequent paper.

To explain the Hilbert modular surface case in [9], let E be a non-biquadratic quartic CM number field with real quadratic subfield $F = \mathbb{Q}(\sqrt{D})$ with fundamental discriminant D. Let σ be the non-trivial Galois automorphism of F. Let

$$V := \{A \in M_2(F) : \sigma(A) = A^\iota\} = \{A = \begin{pmatrix} u & b\sqrt{D} \\ \frac{a}{\sqrt{D}} & \sigma(u) \end{pmatrix} : u \in F, a, b \in \mathbb{Q}\}$$

and let

$$L = \{A = \begin{pmatrix} u & b\sqrt{D} \\ \frac{a}{\sqrt{D}} & \sigma(u) \end{pmatrix} : u \in O_F, a, b \in \mathbb{Z}\}.$$

Here $A \mapsto A^\iota$ is the main involution of $M_2(F)$. The group

$$G(\mathbb{Q}) = \mathrm{GSpin}(V)(\mathbb{Q}) = \{g \in \mathrm{GL}_2(F) : \det g \in \mathbb{Q}^\times\}$$

acts on V via $g.A = gA\sigma(g^{-1})$. The Shimura variety X_K is a Hilbert modular surface, and, for suitable choice of K, is isomorphic to $\mathrm{SL}_2(O_F \oplus \partial^{-1})\backslash\mathbb{H}^2$. Now we describe the CM cycle $\mathrm{CM}(E)$ in [9], the locus of abelian surfaces over $\mathbb{C}$ with CM by O_E, as a formal sum of $Z(W)$'s. For a principally polarized CM abelian surface $\mathbf{A} = (A, \kappa, \lambda)$ of CM type (O_E, Σ), let $M = H_1(A, \mathbb{Z})$ with the action of O_E induced by κ and the symplectic form λ induced by the polarization. Define the lattice

$$L(\mathbf{A}) = \{j \in \mathrm{End}(M) : j \circ \kappa(a) = \kappa(\sigma(a)) \circ j, \ a \in O_F, \ j^* = j\}$$

of special endomorphisms of M with $\mathbb{Z}$-quadratic form $Q(j) = j^2$, where j^* is the 'Rosati' involution induced by λ. Let $V(\mathbf{A}) = L(\mathbf{A}) \otimes \mathbb{Q}$. Then one can show that the rank 4 quadratic lattice $(L(\mathbf{A}), Q) \cong (L, \det)$ is independent of the choice of $\mathbf{A}$. On the other hand, let $E^\sharp$ be the reflex field of (E, Σ), and let $F^\sharp = \mathbb{Q}(\sqrt{\tilde{D}})$ be the real quadratic subfield of $E^\sharp$. It turns out ([17], see also Section 6) that $V(\mathbf{A})$ has a natural $E^\sharp$-vector space structure together with an $F^\sharp$-valued quadratic form $Q^\sharp_{\mathbf{A}}$ such that

$$\mathrm{N}_\Sigma(r) \bullet j = \kappa(r) \circ j \circ \kappa(\bar{r})$$

for any $r \in E$, and

$$\mathrm{tr}_{F^\sharp/\mathbb{Q}} Q^\sharp_{\mathbf{A}}(j) = Q(j), \quad j \in V(\mathbf{A}).$$

Let $W(\mathbf{A}) = (V(\mathbf{A}), Q^\sharp_{\mathbf{A}})$ be the resulting 2-dimensional quadratic space over $F^\sharp$. Then the rational torus associated to $W(\mathbf{A})$ is

$$T_E(R) = \{r \in (R \otimes_{\mathbb{Q}} E)^\times : r\bar{r} \in R^\times\}$$

and its rational points $T_E(\mathbb{Q})$ act on $W(\mathbf{A})$ via $r \bullet j = \frac{1}{r\bar{r}} \kappa(r) \circ j \circ \kappa(\bar{r})$. In Section 6, we will show that the CM cycle $Z(W(\mathbf{A}))$ is naturally equal to

$$Z(\mathbf{A}) = Z(T_E, \mathbf{A}) + Z(T_E, \iota(\mathbf{A})) + Z(T_E, \eta(\mathbf{A})) + Z(T_E, \iota\eta(\mathbf{A})).$$

Here for a principally polarized CM abelian variety $\mathbf{B}$ with CM by O_E, we write $Z(T_E, \mathbf{B})$ for the $C(T_E) = T_E(\mathbb{Q})\backslash T_E(\hat{\mathbb{Q}})$-orbit of $\mathbf{B}$ in X_K. We refer to Section 6 for the action of $C(T_E)$ on CM abelian varieties. Furthermore $\iota(\mathbf{A})$ is the 'complex conjugation' of $\mathbf{A}$, and $\eta \in \mathrm{Aut}(\mathbb{C})$ such that $\eta(\Sigma)$ is another CM type of E different from Σ or $\iota(\Sigma)$. Theorem 1.1 gives a formula for $\Phi(Z(\mathbf{A}), f)$. Notice that $W(\mathbf{A})$ depends on the choice of the $C(T_E)$-orbit of $\mathbf{A}$ in general, and that the set $\mathrm{CM}(E)$ of principally polarized abelian CM surfaces is a finite union of $Z(\mathbf{A})$. So the CM value $\Phi(\mathrm{CM}(E), f)$ is related to a few, not just one, Eisenstein series and L-functions (see Corollary 6.9). When $D \equiv 1 \mod 4$ is a prime, however, the formula becomes simple and only one incoherent Eisenstein series is involved, as in [9] (see Proposition 6.11). We have the following result.

Theorem 1.2. Let E be a CM quartic field with discriminant $D^2\tilde{D}$ with $D \equiv 1 \mod 4$ prime and $\tilde{D} \equiv 1 \mod 4$ square free and with real quadratic subfield $F = \mathbb{Q}(\sqrt{D})$. Let $f \in H_{0,\bar{\rho}_L}$ as above. Then

$$\Phi(\mathrm{CM}(E), f) = \frac{\deg(\mathrm{CM}(E))}{2\Lambda(0,\chi)} \left(\mathrm{CT}[\langle f^+, \mathcal{E}(\tau, L^\sharp)\rangle] - \mathcal{L}'(0, \xi(f), L^\sharp)\right).$$

Here $L^\sharp = O_{E^\sharp}$ with $F^\sharp$-quadratic form $Q^\sharp(r) = -\frac{1}{\sqrt{\tilde{D}}} r\bar{r}$, and $\Lambda(s,\chi)$ is the complete L-function of the quadratic Hecke character χ of F associated to E/F defined in (4.6). □

We expect the factor $\frac{\deg(\mathrm{CM}(E))}{2\Lambda(0,\chi)}$ to be 1 and prove it in some special cases in Section 6. We also give a scalar modular form version of this theorem in Section 6.5. In particular, when f is weakly holomorphic, and $\tilde{D}$ is prime, this theorem recovers the main result in [9], where it was proved using a different method ([9, Theorem 1.4]).

The idea of constructing big CM cycles was communicated to one of the authors (T.Y.) a couple of years ago by Eyal Goren in a private conversation. We thank him for sharing his idea. A slightly more general type of CM point is discussed in [35, Section 5], and our result (Theorem 1.1) can undoubtedly be extended to that case.

It is interesting to note that the Shimura variety $\mathrm{Sh}(G, \mathbb{D})$ attached to $G = \mathrm{GSpin}(V)$ is of PEL-type only for small values of d where accidental isomorphisms occur. In these cases, the moduli theoretic interpretation of the 0-cycles defined in Section 2 is slightly subtle. Thus, for example, as shown in Section 6, in the Hilbert modular surface case, the 0-cycle associated to abelian surfaces with CM by a non-biquadratic quartic CM field E/F is a union of the 0-cycles constructed in Section 3 for the reflex field $E^\sharp/F^\sharp$.

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2 The Shimura variety and its special points

As in Section 1, let F be a totally real number field of degree $d+1$ over $\mathbb{Q}$ with embeddings $\{\sigma_j\}_{j=0}^{d}$ into $\mathbb{R}$. Let W, $(\ ,\ )_W$ be a quadratic space over F of dimension 2 with signature

$$\mathrm{sig}(W) = ((0,2), (2,0), \dots, (2,0)).$$

Let $V = \mathrm{Res}_{F/\mathbb{Q}} W$ be the underlying rational vector space with bilinear form $(x,y)_V = \mathrm{tr}_{F/\mathbb{Q}}(x,y)_W$. There is an orthogonal direct sum

$$V \otimes_{\mathbb{Q}} \mathbb{R} = \oplus_j W_{\sigma_j} \tag{2.1}$$

of real quadratic spaces where $W_{\sigma_j} = W \otimes_{F,\sigma_j} \mathbb{R}$, and $\mathrm{sig}(V) = (2d, 2)$. Let $G = \mathrm{GSpin}(V)$. Then there is a homomorphism

$$\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GSpin}(W) \longrightarrow G \tag{2.2}$$

of algebraic groups over $\mathbb{Q}$ which, on real points, gives the homomorphism

$$\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GSpin}(W)(\mathbb{R}) = \prod_j \mathrm{GSpin}(W_{\sigma_j}) \longrightarrow \mathrm{GSpin}(V \otimes_{\mathbb{Q}} \mathbb{R}) = G(\mathbb{R}), \tag{2.3}$$

associated to the decomposition (2.1).

Lemma 2.1. Let T be the inverse image in G of the subgroup $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(W)$ of $\mathrm{SO}(V)$. Then T is a maximal torus of G and is the image of the homomorphism (2.2). □

Note that there is thus an exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow T \longrightarrow \operatorname{Res}_{F/\mathbb{Q}} \mathrm{SO}(W) \longrightarrow 1 \tag{2.4}$$

of algebraic groups over $\mathbb{Q}$, where $\mathbb{G}_m$ is the kernel of the homomorphism $\mathrm{GSpin}(V) \to \mathrm{SO}(V)$.

A more explicit description of T can be given as follows. The even part $C_F^0(W) = E$ of the Clifford algebra of W over F is a CM field of degree $2d+2$ over $\mathbb{Q}$. The odd part of the Clifford algebra $C_F^1(W) = W = Ew_0$ is a one dimensional vector space over E with quadratic form $Q_W(aw_0) = \alpha \mathrm{N}_{E/F}(a)$, where $\alpha = Q_W(w_0) \in F^\times$ is an element with $\sigma_0(\alpha) < 0$ and $\sigma_j(\alpha) > 0$ for $j \ge 1$. Then, on rational points, we have

$$\begin{array}{ccccc} \operatorname{Res}_{F/\mathbb{Q}} \mathrm{GSpin}(W)(\mathbb{Q}) & \longrightarrow & T(\mathbb{Q}) & \longrightarrow & \operatorname{Res}_{F/\mathbb{Q}} \mathrm{SO}(W)(\mathbb{Q}) \\ \| & & \| & & \| \\ E^\times & \longrightarrow & E^\times/F^1 & \longrightarrow & E^\times/F^\times \end{array}$$

where $E^\times/F^\times \simeq E^1$, via $\beta \mapsto \beta/\bar\beta$ is the kernel of $\mathrm{N}_{E/F}$, and F^1 is the kernel of $N_{F/\mathbb{Q}}$.

Fixing an identification $\mathbb{S} = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \simeq \mathrm{GSpin}(W_{\sigma_0})$, we obtain a homomorphism $h_0 : \mathbb{S} \to G_\mathbb{R}$ of algebraic groups over $\mathbb{R}$ corresponding to the inclusion in the first factor in (2.3). Let $\mathbb{D}$ be the $G(\mathbb{R})$-conjugacy class of h_0. Let $\{e_0, f_0\}$ be a standard basis of $W_0 \subset V \otimes_\mathbb{Q} \mathbb{R}$, i.e., $(e_0, e_0) = (f_0, f_0) = -1$ and $(e_0, f_0) = 0$. Then it is easy to check that teh map

$$g h_0 g^{-1} \mapsto \mathbb{R} g e_0 + \mathbb{R} g f_0$$

gives a bijection between $\mathbb{D}$ and the set of oriented negative 2-planes in $V \otimes_\mathbb{Q} \mathbb{R}$. We will not distinguish between the two interpretations of $\mathbb{D}$. Note that the choice of orientation determined by $\{e_0, f_0\}$ is equivalent to the choice of an extension of σ_0 to an embedding of E into $\mathbb{C}$, which we also denote by σ_0.

Let K be a compact open subgroup of $G(\hat{\mathbb{Q}})$, where $\hat F$ stands for the finite adeles of a number field F. Let $X_K = \mathrm{Sh}(G, h_0)_K$ be the canonical model of the Shimura variety over $\mathbb{Q}$ with

$$X_K(\mathbb{C}) = G(\mathbb{Q}) \backslash \big(\mathbb{D} \times G(\hat{\mathbb{Q}})/K \big).$$

By construction, the homomorphism h_0 factors through $T_\mathbb{R}$ and is fixed by conjugation by $T(\mathbb{R})$, so we have, for any $g \in G(\hat{\mathbb{Q}})$, a special 0-cycle in X_K according to [27, Page 325]

$$Z(T, h_0, g)_K(\mathbb{C}) = T(\mathbb{Q}) \backslash \big(\{h_0\} \times T(\hat{\mathbb{Q}})/K_T^g \big) \to X_K, \qquad [h_0, t] \mapsto [h_0, tg] \tag{2.5}$$

where $K_T^g = T(\hat{\mathbb{Q}}) \cap gKg^{-1}$. Note that K_T^g depends only on the image of g in $\mathrm{SO}(V)(\hat{\mathbb{Q}})$. We will usually drop the subscript K and identify $Z(T, h_0, g)$ with its image in X_K, but every point in $Z(T, h_0, g)$ is counted with multiplicity $\frac{2}{w_{K,T,g}}$ and $w_{K,T,g} = \sharp(T(\mathbb{Q}) \cap gKg^{-1})$. In particular, for a function f on X_K, we have

$$f(Z(T, h_0, g)) = \frac{2}{w_{K,T,g}} \sum_{t \in T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K_T^g} f(h_0, tg). \tag{2.6}$$

When $g = 1$, we will further abbreviate notation and write $Z(T, h_0)$ for $Z(T, h_0, 1)$.

The 0-cycle $Z(T, h_0)$ is defined over $\sigma_0(E)$, the reflex field of (T, h_0). We next describe its Galois conjugates $\tau(Z(T, h_0))$ for $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$.

For $j \in \{0, \dots, d\}$, let $W(j)$ be the unique (up to isomorphism) quadratic space over F such that $W(j) \otimes_F F_v$ and $W \otimes_F F_v$ are isometric for all finite place v of F, and such that

$$\mathrm{sig}(W(j)) = ((2,0), \dots, (2,0), \underset{j}{(0,2)}, (2,0) \dots, (2,0)). \tag{2.7}$$

Note that, although the quadratic spaces $W = W(0)$ and $W(j)$ over F are not isomorphic for $j \neq 0$, there is an isomorphism $C_F^0(W(j)) \simeq C_F^0(W) = E$ of their even Clifford algebras. Let $V(j) = \mathrm{Res}_{F/\mathbb{Q}} W(j)$ with bilinear form $(x, y)_{V(j)} = \mathrm{tr}_{F/\mathbb{Q}}(x, y)_{W(j)}$. The signature of $V(j)$ is $(2d, 2)$ and the quadratic spaces $V(j)$ and V are isomorphic. We fix an isomorphism

$$V(j) \xrightarrow{\sim} V \tag{2.8}$$

and hereafter identity $V(j)$ with V. Let $T(j)$ be the preimage of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(W(j)) \subset \mathrm{SO}(V)$ in G and let $h_0(j) : \mathbb{S} \to G_{\mathbb{R}}$ be the homomorphism defined, as above, by an identification of $\mathbb{S}$ with $\mathrm{GSpin}(W(j) \otimes_{F,\sigma_j} \mathbb{R})$. For $g \in G(\hat{\mathbb{Q}})$, the analogue of the construction above yields a special 0-cycle $Z(T(j), h_0(j), g)$ on X_K defined over $\sigma_j(E)$.

We fix an $\hat{F}$-linear isometry

$$\mu_j : W(j)(\hat{F}) \xrightarrow{\sim} W(\hat{F}). \tag{2.9}$$

Noting that there are canonical identifications $W(j)(\hat{F}) = V(j)(\hat{\mathbb{Q}})$ and $W(\hat{F}) = V(\hat{\mathbb{Q}})$, and using the fixed identification of V and $V(j)$, there is a unique element $g_{j,0} \in \mathrm{O}(V)(\hat{\mathbb{Q}})$ such that the diagram

$$\begin{array}{ccc} W(j)(\hat{F}) & \xrightarrow{\mu_j} & W(\hat{F}) \\ || & & || \\ V(\hat{\mathbb{Q}}) & \xrightarrow{g_{j,0}^{-1}} & V(\hat{\mathbb{Q}}) \end{array} \tag{2.10}$$

Modifying the isometry μ_j by an element of $\mathrm{O}(W)(\hat{F})$, if necessary, we can assume that $g_{j,0} \in \mathrm{SO}(V)(\hat{\mathbb{Q}})$. For any element $g_j \in G(\hat{\mathbb{Q}})$ with image $g_{j,0}$ in $\mathrm{SO}(V)(\hat{\mathbb{Q}})$, the finite adele points of the tori $T(j)$ and T are related, as subgroups of $G(\hat{\mathbb{Q}})$, by

$$T(j)(\hat{\mathbb{Q}}) = g_j T(\hat{\mathbb{Q}}) g_j^{-1}, \tag{2.11}$$

and hence

$$K_{T(j)}^{g_j} = g_j K_T g_j^{-1}. \tag{2.12}$$

These relations depend only on the image $g_{j,0}$ of g_j.

The reciprocity laws for the action of $\mathrm{Aut}(\mathbb{C})$ on special points of Shimura varieties [29], [30], [27], yields the following result.

Lemma 2.2. Let the notation be as above and let $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$.
(1) If $\tau = \sigma_j \circ \sigma_0^{-1}$ on $\sigma_0(E)$, then there is a preimage g_j of $g_{j,0}$, unique up to an element of $\mathbb{Q}^\times$, such that

$$\tau(Z(T, h_0)) = Z(T(j), h_0(j), g_j).$$

(2) If $\tau = \rho$ is complex conjugation, then

$$\tau(Z(T, h_0)) = Z(T, h_0^-).$$

Here h_0^- is the map from $\mathbb{S}$ to $G_\mathbb{R}$ induced by $\mathbb{S} \to \mathrm{GSpin}(W_{\sigma_0}), z \mapsto \bar{z}$. □

We will write

$$Z(T(j), h_0^\pm(j), g_j) = Z(T(j), h_0^+(j), g_j) + Z(T(j), h_0^-(j), g_j).$$

We will also write $z_0^\pm(j) \in \mathbb{D}$ for the oriented negative two planes in $V(\mathbb{R})$ associated $h_0^\pm(j)$. Let

$$Z(W) = \sum_{j=0}^{d} Z(T(j), z_0^\pm(j), g_j) \in Z^{2d}(X_K) \tag{2.13}$$

Then $Z(W)$ is a 0-cycle defined over $\mathbb{Q}$.

3 Special divisors and automorphic Green functions

In this section, we briefly review the special divisors defined in [19] and their 'automorphic' Green functions defined by the first author and Funke using regularized theta liftings [4], [6]. We prove that these special cycles do not intersect with the special cycles defined in Section 2.

Let $x \in V(\mathbb{Q})$ be a vector of positive norm. We write V_x for the orthogonal complement of x in V and G_x for the stabilizer of x in G. So $G_x \cong \mathrm{GSpin}(V_x)$. The sub-Grassmannian

$$\mathbb{D}_x = \{z \in \mathbb{D};\ z \perp x\} \tag{3.1}$$

defines an analytic divisor of $\mathbb{D}$. For $g \in G(\widehat{\mathbb{Q}})$ we consider the natural map

$$G_x(\mathbb{Q})\backslash\mathbb{D}_x \times G_x(\widehat{\mathbb{Q}})/(G_x(\widehat{\mathbb{Q}}) \cap gKg^{-1}) \longrightarrow X_K, \quad (z, g_1) \mapsto (z, g_1 g). \tag{3.2}$$

Its image defines a divisor $Z(x, g)$ on X_K, which is rational over $\mathbb{Q}$. For $m \in \mathbb{Q}_{>0}$ and $\varphi \in S(V(\widehat{\mathbb{Q}}))^K$, if there is an $x_0 \in V(\mathbb{Q})$ with $Q(x_0) = m$, we define the weighted cycle

$$Z(m, \varphi) = \sum_{g \in G_{x_0}(\widehat{\mathbb{Q}})\backslash G(\widehat{\mathbb{Q}})/K} \varphi(g^{-1}x_0) Z(x_0, g). \tag{3.3}$$

It is a divisor on X_K with complex coefficients. Note that, since φ has compact support in $V(\widehat{\mathbb{Q}})$ and the orbits of K on the compact set $G(\widehat{\mathbb{Q}}) \cdot x_0 \cap \operatorname{supp}(\varphi)$ are open, the sum is finite. If there is no $x_0 \in V(\mathbb{Q})$ such that $Q(x_0) = m$, we set $Z(m, \varphi) = 0$.

Proposition 3.1. Let the notation be as above. Then $Z(m, \varphi)$ and $Z(T(j), h_0^{\pm}(j), g_j)$ do not intersect in X_K. □

Proof. It suffices to show that $Z(x, g_1) \cap Z(T, h_0, g_2)$ is empty for every $x \in V(\mathbb{Q})$ with $Q(x) > 0$ and $g_1, g_2 \in G(\widehat{\mathbb{Q}})$. Suppose $P = [z, hg_1] = [z_0, tg_2]$ is in the intersection, where $z_0 = \mathbb{R}e_0 + \mathbb{R}f_0$ is the negative two plane associated to h_0, $z \in \mathbb{D}_x$ is a negative two-plane in $V(\mathbb{R})$ which is orthogonal to x and $h \in G_x(\widehat{\mathbb{Q}})$. Then there are $\gamma \in G(\mathbb{Q})$ and $k \in K$ such that

$$(\gamma)_\infty z = z_0, \quad \hat{\gamma} h g_1 k = t g_2.$$

Here $\hat{\gamma}$ is the image of γ in $G(\widehat{\mathbb{Q}})$. Let $y = \gamma x \in V(\mathbb{Q})$. Then $x \perp z$ implies that $y \perp z_0$, i.e., $(\sigma_0(y), e_0) = (\sigma_0(y), f_0) = 0$. This implies that $\sigma_0(y) = 0$. But $\sigma_0(y)$ is just the projection of y to the first summand of

$$W \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_{j=0}^{d} W \otimes_{F, \sigma_j} \mathbb{R}.$$

Since this map is injective on rational points and $y = \gamma x \neq 0$, we must have $\sigma_0(y) \neq 0$. Thus no such point P can exist. ■

Let L be an even integral lattice in V, i.e., $Q(x) = \frac{1}{2}(x, x) \in \mathbb{Z}$ for $x \in L$, and let

$$L' = \{y \in V : (x, y) \in \mathbb{Z}, \text{ for } x \in L\} \supset L$$

be its dual. For $\mu \in L'/L$, we write $\varphi_\mu = \operatorname{char}(\mu + \hat{L}) \in S(V(\widehat{\mathbb{Q}}))$ and $Z(m, \mu) = Z(m, \varphi_\mu)$, where $\hat{L} = L \otimes \widehat{\mathbb{Z}}$. Associated to the reductive dual pair $(\mathrm{SL}_2, \mathrm{O}(V))$ there is a Weil representation $\omega = \omega_\psi$ of $\mathrm{SL}_2(\mathbb{A})$ on the

Schwartz space $S(V(\mathbb{A}))$, where ψ is the 'canonical' unramified additive character of $\mathbb{Q}\backslash\mathbb{A}$ with $\psi_\infty(x) = e(x)$. Since the subspace $S_L = \oplus\mathbb{C}\varphi_\mu \subset S(V(\widehat{\mathbb{Q}}))$ is preserved by the action of $\mathrm{SL}_2(\widehat{\mathbb{Z}})$, there is a representation ρ_L of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ on this space defined by the formula

$$\rho_L(\gamma)\varphi = \bar{\omega}(\hat{\gamma})\varphi := \overline{\omega(\hat{\gamma})\bar{\varphi}}, \tag{3.4}$$

where $\hat{\gamma} \in \mathrm{SL}_2(\hat{\mathbb{Z}})$ is the image of γ. This representation is given explicitly by Borcherds as

$$\rho_L(T)(\varphi_\mu) = e(Q(\mu^2))\,\varphi_\mu, \tag{3.5}$$

$$\rho_L(S)(\varphi_\mu) = \frac{e((1-d)/4)}{\sqrt{|L'/L|}} \sum_{\nu\in L'/L} e(-(\mu,\nu))\,\varphi_\nu, \tag{3.6}$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, see e.g. [1], [20], [4]. Note that, by (3.4), the complex conjugate $\bar{\rho}_L$ is just the restriction of ω to the subgroup $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\widehat{\mathbb{Z}})$.

Recall that a smooth function $f : \mathbb{H} \to S_L$ is called a *harmonic weak Maass form* (of weight k with respect to Γ and ρ_L) if it satisfies:

(i) $f\,|_{k,\rho_L}\,\gamma = f$ for all $\gamma \in \Gamma$; i.e.,

$$f(\gamma\tau) = (c\tau + d)^k \rho_L(\gamma) f(\tau).$$

(ii) there is a S_L-valued Fourier polynomial

$$P_f(\tau) = \sum_{\mu\in L'/L} \sum_{n\le 0} c^+(n,\mu)\, q^n\, \varphi_\mu$$

such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$;

(iii) $\Delta_k f = 0$, where

$$\Delta_k := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

is the usual weight k hyperbolic Laplace operator (see [6]).

The Fourier polynomial P_f is called the *principal part* of f. We denote the vector space of these harmonic weak Maass forms by H_{k,ρ_L}. Any weakly holomorphic modular form is a harmonic weak Maass form. The Fourier expansion of any $f \in H_{k,\rho_L}$ gives a unique decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c^+(n,\mu)\, q^n\, \varphi_\mu, \tag{3.7a}$$

$$f^-(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n < 0}} c^-(n,\mu)\, \Gamma(1-k, 4\pi|n|v)\, q^n\, \varphi_\mu, \tag{3.7b}$$

and, for $a > 0$, $\Gamma(s,a) = \int_a^\infty e^{-t} t^{s-1}\, dt$ is the incomplete Γ-function. We refer to f^+ as the *holomorphic part* and to f^- as the *non-holomorphic part* of f.

Recall that there is an antilinear differential operator $\xi = \xi_k : H_{k,\rho_L} \to S_{2-k,\bar\rho_L}$, defined by

$$f(\tau) \mapsto \xi(f)(\tau) := 2iv^k \overline{\frac{\partial}{\partial\bar\tau} f(\tau)}. \tag{3.8}$$

By [6, Corollary 3.8], one has the exact sequence

$$0 \longrightarrow M^!_{k,\rho_L} \longrightarrow H_{k,\rho_L} \xrightarrow{\ \xi\ } S_{2-k,\bar\rho_L} \longrightarrow 0\ . \tag{3.9}$$

Let $f \in H_{1-d,\bar\rho_L}$ be a harmonic weak Maass form of weight $1-d$ with representation $\bar\rho_L$ for Γ, and denote its Fourier expansion as in (3.7). Let $S_L^\vee$ be the dual space of S_L—the space of linear functionals on S_L, and let $\{\varphi_\mu^\vee\}$ be the dual basis in $S_L^\vee$ of the basis $\{\varphi_\mu\}$ of S_L. Recall that the Siegel theta function

$$\theta_L(\tau, z, g) = \sum_\mu \theta(\tau, z, g, \varphi_\mu)\, \varphi_\mu^\vee$$

is an $S_L^\vee$-valued holomorphic modular form of weight $d-1$ for Γ and ρ_L defined as follows (see [10, Section 2] or [20] for details). For $z \in \mathbb{D}$, one has decomposition

$$V(\mathbb{R}) = z \oplus z^\perp, \quad x = x_z + x_{z^\perp}.$$

Let $(x,x)_z = -(x_z, x_z) + (x_{z^\perp}, x_{z^\perp})$ and define the associated Gaussian by

$$\varphi_\infty(x,z) = e^{-\pi(x,x)_z}. \tag{3.10}$$

Then, for $\tau \in \mathbb{H}$, $[z,g] \in X_K$, and $\varphi \in S(V(\hat{\mathbb{Q}}))$, the theta function is given by

$$\theta(\tau, z, g, \varphi) = v^{\frac12(1-d)} \sum_{x \in V(\mathbb{Q})} \omega(g_\tau)\varphi_\infty(x,z)\varphi(g^{-1}x), \qquad g_\tau = \begin{pmatrix} v^{\frac12} & uv^{-\frac12} \\ & v^{-\frac12} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Here g acts on V via its image in $\mathrm{SO}(V)$.

We consider the regularized theta integral

$$\Phi(z, g, f) = \int_{\mathcal{F}}^{reg} \langle f(\tau), \theta_L(\tau, z, g)\rangle \, d\mu(\tau) \tag{3.11}$$

for $z \in \mathbb{D}$ and $g \in G(\widehat{\mathbb{Q}})$, where $\mathcal{F}$ is the standard domain for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. The integral is regularized as in [1], [6], that is, $\Phi(z, g, f)$ is defined as the constant term in the Laurent expansion at $s = 0$ of the function

$$\lim_{T\to\infty} \int_{\mathcal{F}_T} \langle f(\tau), \theta_L(\tau, z, g)\rangle \, v^{-s} d\mu(\tau). \tag{3.12}$$

Here $\mathcal{F}_T = \{\tau \in \mathbb{H};\ |u| \le 1/2,\ |\tau| \ge 1, \text{ and } v \le T\}$ denotes the truncated fundamental domain and the integrand

$$\langle f(\tau), \theta_L(\tau, z, g)\rangle = \sum_{\mu \in L'/L} f_\mu(\tau)\theta(\tau, z, g, \varphi_\mu) \tag{3.13}$$

is the pairing of f with the Siegel theta function, viewed as a linear functional on the space S_L.

The following theorem summarizes some properties of the function $\Phi(z, g, f)$ in the setup of the present paper (see [4], [6]).

Theorem 3.2. The function $\Phi(z, g, f)$ is smooth on $X_K \backslash Z(f)$, where

$$Z(f) = \sum_{\mu \in L'/L} \sum_{m>0} c^+(-m, \mu) Z(m, \mu). \tag{3.14}$$

It has a logarithmic singularity along the divisor $-2Z(f)$. The $(1,1)$-form $dd^c\Phi(z, g, f)$ can be continued to a smooth form on all of X_K. We have the Green current equation

$$dd^c[\Phi(z, g, f)] + \delta_{Z(f)} = [dd^c\Phi(z, g, f)], \tag{3.15}$$

where δ_Z denotes the Dirac current of a divisor Z. Moreover, if Δ_z denotes the invariant Laplace operator on $\mathbb{D}$, normalized as in [4], we have

$$\Delta_z \Phi(z, g, f) = \frac{d}{2} \cdot c^+(0, 0). \tag{3.16}$$

□

In particular, the theorem implies that $\Phi(z, g, f)$ a Green function for the divisor $Z(f)$ in the sense of Arakelov geometry in the normalization of [38]. More precisely, by the results of Borcherds [1] and Bruinier [4], [6], $\Phi(f) = -\log ||\Psi(f)||^2$, for a meromorphic section $\Psi(f)$ of the line bundle $\mathcal{L}^{\frac{1}{2}c^+(0,0)}$, where the metrized line bundle $(\mathcal{L}, ||\ ||)$ on X_K is defined in [20]. If $c^+(0, 0) = 0$, then the meromorphic function $\Psi(f)$ extends to a smooth compactification of X_K. The line bundle $\mathcal{L}$ also has an extension to such a compactification, but the metric becomes singular along the divisor at infinity. Thus, if the constant term $c^+(0,0)$ of f does not vanish, one actually has to work with the generalization of Arakelov geometry given in [8]. When $c^+(0,0) = 0$, $\Phi(z, g, f)$ is harmonic, and we refer to it as the *automorphic Green function* associated with $Z(f)$. Notice also that $Z(f)$ has coefficients in $\mathbb{Q}(f)$, the field generated by the $c^+(-m, \mu)$, $m > 0$.

4 CM values of Siegel theta functions and Eisenstein series

Recall that, for each j, we have fixed an isomorphism $V \simeq V(j) = \mathrm{Res}_{F/\mathbb{Q}} W(j)$ of rational quadratic spaces, and hence an identification

$$S(V(\mathbb{A}_\mathbb{Q})) = S(W(j)(\mathbb{A}_F)), \qquad \varphi \mapsto \varphi_{F,j} \tag{4.1}$$

of the corresponding Schwartz spaces. For example, if $\varphi_F = \otimes_w \varphi_{F,w} \in S(W(j)(\mathbb{A}_F))$, with w running over the places of F, then the corresponding $\varphi \in S(V(\mathbb{A}_\mathbb{Q}))$ is also factorizable, with local component $\varphi_v = \otimes_{w|v} \varphi_{F,w}$ in the space

$$S(\mathrm{Res}_{F/\mathbb{Q}} W(j)(\mathbb{Q}_v)) = S(\oplus_{w|v} W(j)(F_w)) = \otimes_{w|v} S(W(j)(F_w)).$$

These identifications are compatible with the Weil representations of $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$ and $\mathrm{SL}_2(\mathbb{A}_F)$ for our fixed additive character ψ of $\mathbb{A}_\mathbb{Q}$ and the character $\psi_F = \psi \circ \mathrm{tr}_{F/\mathbb{Q}}$ of $\mathbb{A}_F$, i.e.,

$$\omega_{V,\psi}(g')\varphi = \omega_{W(j),\psi_F}(g')\varphi_{F,j},$$

where, on the right side, we view $g' \in \mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$ as an element of $\mathrm{SL}_2(\mathbb{A}_F)$. We write φ_F for $\varphi_{F,0}$. Moreover, we will frequently abuse notation and write φ for φ_F and identify $S(W(\mathbb{A}_F))$ with $S(V(\mathbb{A}))$. Note that the Weil representations $\omega_{W(j),\psi_F}$ of $\mathrm{SL}_2(\mathbb{A}_F)$, which are now all realized on $S(V(\mathbb{A}_\mathbb{Q}))$, via (4.1), do not coincide in general. The point is that the group $\mathrm{SL}_2(F)$ in the dual pair $(\mathrm{SL}_2(F), \mathrm{Res}_{F/\mathbb{Q}} \mathrm{O}(W(j)))$ arises as the commutant in the ambient symplectic group of the subgroup $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{O}(W(j)) \subset \mathrm{O}(V)$, i.e., by a seesaw construction, and these subgroups do not coincide.

Recall that, for each j, we have fixed an isometry $\mu_j : W(j)(\hat{F}) \xrightarrow{\sim} W(\hat{F})$, and an element $g_{j,0} \in \mathrm{SO}(V)(\hat{\mathbb{Q}})$ so that the diagram (2.10) commutes.

Lemma 4.1. (i) For any $\varphi \in S(V(\hat{\mathbb{Q}}))$, recall that $\varphi_{F,0} = \varphi_F$ is identified with ϕ via $S(W(\hat{F})) \cong S(V(\hat{\mathbb{Q}}))$. Then

$$\mu_j^*(\varphi) = (\omega(g_{j,0})\varphi)_{F,j},$$

where $g \in \mathrm{SO}(V)(\hat{\mathbb{Q}})$ acts on $S(V(\hat{\mathbb{Q}}))$ by $(\omega(g)\varphi)(x) = \varphi(g^{-1}x)$.
(ii) The map $\mu_j^* : S(W(\hat{F})) \to S(W(j)(\hat{F}))$ intertwines the Weil representations ω_{W,ψ_F} and $\omega_{W(j),\psi_F}$ of $\mathrm{SL}_2(\hat{\mathbb{Q}})$ on these spaces.

(iii) For $g' \in \mathrm{SL}_2(\hat{\mathbb{Q}})$, and $\varphi \in S(V(\hat{\mathbb{Q}}))$,

$$\omega_{W(j),\psi_F}(g')\omega(g_{j,0})\varphi = \omega(g_{j,0})\,\omega_{W,\psi_F}(g')\varphi.$$

□

Here in parts (i) and (iii), we are working in the fixed space $S(V(\hat{\mathbb{Q}}))$ with natural linear action of $g \in \mathrm{SO}(V)(\hat{\mathbb{Q}})$ defined in (i) and the *various* Weil representation actions of $\mathrm{SL}_2(\hat{F})$, as described above.

For $z \in \mathbb{D}$, the Gaussian $\varphi_\infty(\cdot, z) \in S(V(\mathbb{R}))$ is defined by (3.10). The points $z_0^\pm(j) \in \mathbb{D}$ are the fixed points of $T(j)(\mathbb{R})$, and

$$\varphi_\infty(\cdot, z_0^\pm(j)) = \otimes_i \varphi_{\infty,W(j)_{\sigma_i}},$$

in

$$S(V(\mathbb{R})) = S(\mathrm{Res}_{F/\mathbb{Q}}(W(j))(\mathbb{R})) = \otimes_i S(W(j)_{\sigma_i}),$$

where $W(j)_{\sigma_i} = W(j) \otimes_{F,\sigma_i} \mathbb{R}$, and

$$\varphi_{\infty,W(j)_{\sigma_i}}(x) = e^{-\pi|(x,x)_{W(j)_{\sigma_i}}|}$$

is the Gaussian for the definite space $W(j)_{\sigma_i}$. Note that $\varphi_{\infty,W(j)_{\sigma_i}}$ is $\mathrm{SO}(W_{j,\sigma_i})$-invariant, and is an eigenfunction of $\mathrm{SO}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{R})$ with respect to the Weil representation $\omega_{W_{j,\sigma_i}}$ of 'weight' $+1$ for $i \neq j$ and -1 for $i = j$.

For a K-invariant Schwartz function $\varphi \in S(V(\hat{\mathbb{Q}}))^K$ and $\tau \in \mathbb{H}$, the theta function

$$\theta(\tau, z, g, \varphi) = v^{\frac{1-d}{2}} \sum_{x \in V(\mathbb{Q})} \omega_V(g_\tau)\varphi_\infty(x, z)\varphi(g^{-1}x) \tag{4.2}$$

is an automorphic function of $[z, g] \in X_K$, where $z \in \mathbb{D}$ and $g \in G(\hat{\mathbb{Q}})$. By the preceding discussion, the pullback of this function to $Z(T(j), z_0^\pm(j), g_j)$ coincides with the pullback of the Hilbert theta function associated to the quadratic space $W(j)$,

$$\theta(\vec{\tau}, t, (\omega(g_{j,0})\varphi)_{F,j}) = v_j\, N(\vec{v})^{-\frac{1}{2}} \sum_{x \in W(j)(F)} \omega_{W(j)}(g_{\vec{\tau}})\varphi_{\infty,W(j)}(x)\, (\omega(g_{j,0})\varphi)_{F,j}(t^{-1}x), \tag{4.3}$$

via the diagonal embedding of $\mathbb{H}$ into $\mathbb{H}^{d+1}$. Here $\vec{\tau} \in \mathbb{H}^{d+1}$, with components $\tau_r = u_r + iv_r$, $N(\vec{v}) = \prod_r v_r$, and $g_{\vec{\tau}} \in \mathrm{SL}_2(\mathbb{R})^{d+1}$ with component g_{τ_r} in the rth slot. This theta function has weight

$$\mathbf{1}(j) := (1, \dots, -1, \dots, 1),$$

with -1 in the jth slot.

Let $\chi = \chi_{E/F}$ be the quadratic Hecke character of F associated to E/F, and let $I(s, \chi) = \otimes_v I(s, \chi_v)$ be the representation of $\mathrm{SL}_2(\mathbb{A}_F)$ induced from the character $\chi\,|\;|^s$ of the standard Borel subgroup. We write $\Phi^k_{\sigma_i}$

for the unique eigenfunction of $\mathrm{SO}_2(\mathbb{R}) \subset \mathrm{SL}_2(F \otimes_{F,\sigma_i} \mathbb{R})$ in $I(s, \chi_{\sigma_i})$ of weight k with $\Phi^k_{\sigma_i}(1, s) = 1$. We define sections in $I_\infty(s, \chi_\infty) = \otimes_i I(s, \chi_{\sigma_i})$ by

$$\Phi^{\mathbf{1}}_\infty(s) = \otimes_i \Phi^1_{\sigma_i}(s),$$

and

$$\Phi^{\mathbf{1}(j)}_\infty(s) = \Phi^{-1}_{\sigma_j}(s) \otimes (\otimes_{i \neq j} \Phi^1_{\sigma_i}(s)).$$

For each j, there is an $\mathrm{SL}_2(\hat{F})$-equivariant map

$$\lambda_j : S(W(j)(\hat{F})) \to I_f(0, \chi_f), \qquad \varphi \mapsto \lambda_j(\varphi)(g) = \omega_{W(j),\psi_F}(g)\varphi(0).$$

By (ii) of Lemma 4.1, these maps for various j's are related as follows.

Lemma 4.2. For $\varphi \in S(V(\hat{\mathbb{Q}}))$, one has

$$\lambda_j(\mu_j^*(\varphi_F)) = \lambda_0(\varphi_F)$$

□

Let $\Phi_\varphi(s) \in I_f(s, \chi_f)$ be the unique standard section with $\Phi_\varphi(g, 0) = \lambda_0(\varphi) = \lambda_j(\mu_j^*(\varphi))$. For $\varphi \in S(W(\hat{F})) = S(V(\hat{\mathbb{Q}}))$ and $\vec{\tau} = (\tau_0, \ldots, \tau_d) \in \mathbb{H}^{d+1}$ with $\tau_r = u_r + iv_r$,, we define the Hilbert-Eisenstein series

$$E(\vec{\tau}, s, \varphi, \mathbf{1}) = N(\vec{v})^{-\frac{1}{2}} E(g_{\vec{\tau}}, s, \Phi^{\mathbf{1}}_\infty \otimes \Phi_\varphi) \tag{4.4}$$

and

$$E(\vec{\tau}, s, \varphi, \mathbf{1}(j)) = v_j\, N(\vec{v})^{-\frac{1}{2}} E(g_{\vec{\tau}}, s, \Phi^{\mathbf{1}(j)}_\infty \otimes \Phi_\varphi). \tag{4.5}$$

Here $N(\vec{v}) = \prod_r v_r$. Note that, $\Phi^{\mathbf{1}(j)}_\infty(s)$ is associated to the Gaussian $\varphi_{\infty,W(j)}$, so that $E(\vec{\tau}, s, \varphi, \mathbf{1}(j))$ is a coherent Eisenstein series of weight $\mathbf{1}(j)$ attached to the function $\varphi_{\infty,W(j)} \otimes \mu_j^*(\varphi) \in S(W(j)(\mathbb{A}_F))$ and $E(\vec{\tau}, s, \varphi, \mathbf{1})$ is an incoherent Eisenstein series of parallel weight $\mathbf{1}$ (independent of j). The two Eisenstein series are related as follows by an observation of [20, (2.17)], [10, Lemma 2.3],

Lemma 4.3. Write $\bar{\partial}_j = \frac{\partial}{\partial \bar{\tau}_j} d\bar{\tau}_j$. Then

$$-2\bar{\partial}_j \left(E'(\vec{\tau}, 0, \varphi, \mathbf{1})\, d\tau_j\right) = E(\vec{\tau}, 0, \varphi, \mathbf{1}(j))\, d\mu(\tau_j).$$

□

In this paper, we normalize the Haar measure dh on $\mathrm{SO}(W(j))(\mathbb{A}_F)$ so that

$$\mathrm{vol}(\mathrm{SO}(W(j))(F) \backslash \mathrm{SO}(W(j))(\mathbb{A}_F)) = 2,$$

and write $dh = dh_\infty \, dh_f$ where $dh_\infty = \prod_i dh_{\sigma_i}$ with $\mathrm{vol}(\mathrm{SO}(W(j)_{\sigma_i}), dh_{\sigma_i}) = 1$. For the convenience of the reader, we first recall [37, Lemma 2.13].

Lemma 4.4. For any function f on

$$Z(T(j), z_0(j), g_j) = T(j)(\mathbb{Q})\backslash\big(\{z_0(j)\} \times T(j)(\hat{\mathbb{Q}})/K_{T(j)}^{g_j}\big),$$

the weighted sum (2.6) of the values of f over this discrete finite set is given by

$$f(Z(T(j), z_0(j), g_j)) = \frac{1}{2} \deg Z(T, z_0) \int_{\mathrm{SO}(W(j))(F)\backslash \mathrm{SO}(W(j))(\hat{F})} f(z_0(j), t)\, dt.$$

Here

$$\deg Z(T, z_0) = \frac{4}{\mathrm{vol}(K_T)}$$

is independent of j. □

Proof. By [37, Lemma 2.13], the formula holds with $\deg Z(T, z_0)$ replaced by the quantity $2/\mathrm{vol}(K_{T(j)}^{g_j})$. So it suffices to check $\mathrm{vol}(K_{T(j)}^{g_j}) = \mathrm{vol}(K_T)$ is independent of j. But this is immediate by (2.11) and (2.12). ■

Proposition 4.5. With the notation as above,

$$\theta(\tau, Z(T(j), z_0(j), g_j), \varphi) = C \cdot E(\tau^\Delta, 0, \varphi, \mathbf{1}(j))$$

where

$$C = \frac{1}{2} \deg(Z(T, z_0)).$$

□

Proof. Since $\mathrm{vol}(\mathrm{SO}(W(j))_{\sigma_i}) = 1$, one has by Lemma 4.4 that

$$\theta(\tau, Z(T(j), z_0(j), g_j) = \frac{1}{2} \deg Z(T, z_0) \int_{\mathrm{SO}(W(j))(F)\backslash \mathrm{SO}(W(j))(\mathbb{A}_F)} \theta(\tau^\Delta, t, (\omega(g_{j,0})\varphi)_{F,j})\, dt,$$

where the theta function in the integral is given by (4.3). Now the proposition follows from the Siegel-Weil formula. ■

For $\chi = \chi_{E/F}$ as above, let

$$\Lambda(s, \chi) = A^{\frac{s}{2}} (\pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2}))^{d+1} L(s, \chi), \quad A = N_{F/\mathbb{Q}}(\partial_F d_{E/F}) \tag{4.6}$$

be the complete L-function of χ. It is a holomorphic function of s with functional equation

$$\Lambda(s,\chi) = \Lambda(1-s,\chi),$$

and

$$\Lambda(1,\chi) = \Lambda(0,\chi) = L(0,\chi) = 2^{d-\delta}\,\frac{h(E)}{w(E)\,h(F)} \in \mathbb{Q}^\times,$$

where $2^\delta = |\mathcal{O}_E^\times : \mu(E)\mathcal{O}_F^\times|$ is 1 or 2. Let

$$E^*(\vec{\tau}, s, \varphi, \mathbf{1}) = \Lambda(s+1,\chi)\, E(\vec{\tau}, s, \varphi, \mathbf{1})$$

be the normalized incoherent Eisenstein series.

Proposition 4.6. Let $\varphi = \varphi_F \in S(V(\hat{\mathbb{Q}})) = S(W(\hat{F}))$. For a totally positive element $t \in F_+^\times$, let $a(t,\varphi)$ be the t-th Fourier coefficient of $E^{*,\prime}(\vec{\tau}, 0, \varphi, \mathbf{1})$. This coefficient is independent of $\vec{v}$. The constant term of $E^{*,\prime}(\vec{\tau}, 0, \varphi, \mathbf{1})$ has the form

$$\varphi(0)\,\big(\,\Lambda(0,\chi)\log \mathrm{N}(\vec{v}) + a_0(\varphi)\,\big),$$

for a constant $a_0(\varphi)$ depending only on φ. Let

$$\mathcal{E}(\tau,\varphi) = \varphi(0)\,a_0(\varphi) + \sum_{n\in\mathbb{Q}_{>0}} a_n(\varphi)\,q^n$$

where

$$a_n(\varphi) = \sum_{t\in F_+^\times,\ \mathrm{tr}_{F/\mathbb{Q}}\,t=n} a(t,\varphi).$$

Then, writing τ^Δ for the diagonal image of $\tau \in \mathbb{H}$ in $\mathbb{H}^{d+1}$,

$$E^{*,\prime}(\tau^\Delta, 0, \varphi, \mathbf{1}) - \mathcal{E}(\tau,\varphi) - \varphi(0)\,\Lambda(0,\chi)\,(d+1)\,\log v$$

is of exponential decay as v goes to infinity. Moreover, for $n > 0$

$$a_n(\varphi) = \sum_p a_{n,p}(\varphi)\log p$$

with $a_{n,p}(\varphi) \in \mathbb{Q}(\varphi)$, the subfield of $\mathbb{C}$ generated by the values $\varphi(x)$, $x \in V(\hat{\mathbb{Q}})$. □

Proof. Let $\mathcal{C} = \otimes_v \mathcal{C}_v$ be the incoherent collection of local quadratic F_v-spaces with $\mathcal{C}_v = W \otimes_F F_v$ for all finite places v and with $\mathcal{C}_\infty$ is totally positive definite. In particular, $\hat{\mathcal{C}} := \otimes_{v<\infty} \mathcal{C}_v = W(\hat{F})$. Then

$$\Phi_\varphi(0) \otimes \Phi_\infty^{\mathbf{1}}(0) = \lambda(\varphi \otimes \varphi_{\infty,\mathcal{C}})$$

for $\varphi \otimes \varphi_{\infty,\mathcal{C}} \in S(\mathcal{C}) = S(\hat{W}) \otimes S(\mathcal{C}_\infty)$, where $\varphi_{\infty,\mathcal{C}} = \otimes_i \varphi_{\sigma_i}$ is the product of the Gaussians for the positive definite binary quadratic spaces $\mathcal{C}_{\sigma_i}$. Thus $E(\vec{\tau}, s, \varphi, \mathbf{1})$ is an incoherent Eisenstein series according to [18] and $E^*(\vec{\tau}, 0, \varphi, \mathbf{1}) = 0$. By linearity, we may assume that the function $\varphi = \otimes_v \varphi_v \in S(W(\hat{F}))$ is factorizable. Then, the Fourier expansion can be written as

$$E^*(\vec{\tau}, s, \varphi, \mathbf{1}) = E_0^*(\vec{\tau}, s, \varphi, \mathbf{1}) + \sum_{t \in F^\times} E_t^*(\vec{\tau}, s, \varphi, \mathbf{1})$$

with

$$E_t^*(\vec{\tau}, s, \varphi, \mathbf{1}) = A^{\frac{s}{2}} \prod_{\mathfrak{p}<\infty} W_{t,\mathfrak{p}}^*(1, s, \varphi_\mathfrak{p}) \prod_{i=0}^{d} W_{t,\sigma_i}^*(\tau_i, s, \Phi_{\sigma_i}^1)$$

and

$$E_0^*(\vec{\tau}, s, \varphi, \mathbf{1}) = \varphi(0)\Lambda(s+1, \chi)\,\mathrm{N}(\vec{v})^{\frac{s}{2}} + A^{\frac{s}{2}} \prod_{\mathfrak{p}<\infty} W_{0,\mathfrak{p}}^*(1, s, \varphi_\mathfrak{p}) \prod_{i=0}^{d} W_{0,\sigma_i}^*(\tau_i, s, \Phi_{\sigma_i}^1).$$

Here, for $g' \in \mathrm{SL}_2(F_\mathfrak{p})$,

$$W_{t,\mathfrak{p}}^*(g', s, \varphi_\mathfrak{p}) = L_\mathfrak{p}(s+1, \chi_v)\, W_{t,\mathfrak{p}}(g', s, \varphi_\mathfrak{p})$$

and

$$W_{t,\sigma_i}^*(\tau_i, s, \Phi_{\sigma_i}^1) = \pi^{-\frac{s+2}{2}} \Gamma(\frac{s+2}{2})\, v_i^{-\frac{1}{2}}\, W_{t,\sigma_i}(g_{\tau_i}, s, \Phi_{\sigma_i}^1)$$

are the normalized local Whittaker functions, which are computed in [24] and [41] in special cases. In particular, [24, Proposition 2.6] (see also [41, Proposition 1.4]) asserts that[†]

$$\begin{aligned}
&W_{t,\sigma_i}^*(\tau_i, 0, \Phi_{\sigma_i}^1) = 2\,\gamma(\mathcal{C}_{\sigma_i})\, e(\sigma_i(t)\tau_i), && \text{if } \sigma_i(t) > 0,\\
&W_{0,\sigma_i}^*(\tau_i, s, \Phi_{\sigma_i}^1) = \gamma(\mathcal{C}_{\sigma_i})\, v_i^{-s/2} \pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2}), &&\\
&W_{t,\sigma_i}^*(\tau_i, 0, \Phi_{\sigma_i}^1) = 0, && \text{if } \sigma_i(t) < 0.
\end{aligned}$$

Here $\gamma(\mathcal{C}_{\sigma_i})$ is the local Weil index, an 8-th root of unity. Moreover, in the last case,

$$W_{t,\sigma_i}^{*,\prime}(\tau_i, 0, \Phi_{\sigma_i}^1) = \gamma(\mathcal{C}_{\sigma_i})\, e(\sigma_i(t)\tau_i)\, \beta_1(4\pi|\sigma_i(t)|v_i)$$

[†]The extra '$-$' in the formula is due to the fact that we use $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ here for the local Whittaker function instead of w^{-1} in [24].

is of exponential decay when v_i goes to infinity. Here

$$\beta_1(x) = \int_1^\infty e^{-xt} t^{-1}\, dt, \quad x > 0.$$

On the other hand, suppose that everything is unramified at a finite prime $\mathfrak{p}$. This means that $E/\mathbb{Q}$ is unramified at primes over $\mathfrak{p}$, we identify $W_\mathfrak{p} = E_\mathfrak{p}$ with quadratic form $(x,x)_\mathfrak{p} = \alpha N_{E_\mathfrak{p}/F_\mathfrak{p}}(x)$ for $\alpha \in O_{F_\mathfrak{p}}^\times$, and $\varphi_\mathfrak{p} = \mathrm{char}(O_{E_\mathfrak{p}})$. Then, by [41, Proposition 1.1], for $t \neq 0$

$$\gamma(\mathcal{C}_\mathfrak{p})^{-1}\, W_{t,\mathfrak{p}}^*(1, s, \varphi_\mathfrak{p}) = \begin{cases} 0 & \text{if } t \notin O_{F_\mathfrak{p}}, \\ \mathrm{ord}_\mathfrak{p}\, t + 1 & \text{if } \mathfrak{p} \text{ split in } E/F, t \in O_{F_\mathfrak{p}}, \\ \frac{1}{2}(1 + (-1)^{\mathrm{ord}_\mathfrak{p}\, t}) & \text{if } \mathfrak{p} \text{ inert in } E/F, t \in O_{F_\mathfrak{p}}. \end{cases}$$

In general, $\gamma(\mathcal{C}_\mathfrak{p})^{-1} W_{t,\mathfrak{p}}^*(1, s, \varphi_\mathfrak{p})$ is a polynomial of $\mathrm{N}(\mathfrak{p})^{-s}$ with coefficients in $\mathbb{Q}(\varphi_\mathfrak{p})$ ([25]). For $t \neq 0$, let $D(t) = D(t, \mathcal{C})$ be the 'Diff' set of places $\mathfrak{p}$ of F (including infinite places) such that $\mathcal{C}_\mathfrak{p}$ does not represent t, as defined in [18]. Then $D(t)$ is a finite set of odd order, and for every $\mathfrak{p} \in D(t)$, the local Whittaker function at v vanishes at $s = 0$. So $E_t^{*,\prime}(\vec{\tau}, 0, \varphi) = 0$ unless $D(t)$ has exactly one element. Assuming this and restricting $\vec{\tau}$ to the diagonal $\tau^\Delta = (\tau, \cdots, \tau)$ with $\tau = u + \sqrt{-1}v \in \mathbb{H}$, there are two subcases.

When $D(t) = \{\sigma_i\}$ for some i, the above formulae show that

$$E_t^{*,\prime}(\tau^\Delta, 0, \varphi, \mathbf{1}) = W_{t,\sigma_i}^{*,\prime}(\tau, 0, \Phi_{\sigma_i}^1) \prod_{\mathfrak{p} \neq \sigma_i} W_{t,\mathfrak{p}}^*(1, 0, \varphi_\mathfrak{p})$$

is of exponential decay when $v = \mathrm{Im}(\tau) \to \infty$.

When $D(t) = \{\mathfrak{p}\}$ for some finite prime $\mathfrak{p}$, and $t \in F_+^\times$ is totally positive,

$$E_t^{*,\prime}(\tau^\Delta, 0, \varphi, \mathbf{1}) = a(t, \varphi)\, q^{\mathrm{tr}_{F/\mathbb{Q}}\, t}, \quad q = e(\tau)$$

for some $a(t, \varphi) \in \mathbb{Q}(\varphi) \log p$, where p is the prime below $\mathfrak{p}$. Here we have used the fact that

$$\prod_{\mathfrak{p} < \infty} \gamma(\mathcal{C}_\mathfrak{p}) \prod_{i=0}^{d} \gamma(\mathcal{C}_{\sigma_i}) = -1.$$

Finally, for the constant term, one has (see e.g., [41, Section 1] or [25])

$$E_0^*(\vec{\tau}, s, \varphi, \mathbf{1}) = \varphi(0)\big(\ \mathrm{N}(\vec{v})^{\frac{s}{2}}\, \Lambda(s+1, \chi) + \mathrm{N}(\vec{v})^{-\frac{s}{2}} \Lambda(1-s, \chi) M_\varphi(s)\ \big)$$

where $M_\varphi(s)$ is a product of finitely many polynomials in $\mathrm{N}(\mathfrak{p})^{-s}$ for finitely many 'bad' $\mathfrak{p}$, and $M_\varphi(0) = -1$. Recalling that $E_0^*(\tau^\Delta, 0, \varphi, \mathbf{1}) = 0$, this gives, for $\tau \in \mathbb{H}$,

$$E_0^{*,\prime}(\tau^\Delta, 0, \varphi, \mathbf{1}) = \varphi(0)\ \left(\Lambda(1,\chi)(d+1)\log v + 2\Lambda'(1,\chi) + \Lambda(1,\chi)M_\varphi'(0)\right). \tag{4.7}$$

The constant term of $E^{*,\prime}(\tau^\Delta, 0, \varphi, \mathbf{1})$ as a (non-holomorphic) elliptic modular form is

$$E_0^{*,\prime}(\tau^\Delta, 0, \varphi, \mathbf{1}) + \sum_{0 \ne t \in F, \mathrm{tr}_{F/\mathbb{Q}}\, t = 0} E_t^{*,\prime}(\tau^\Delta, 0, \varphi, \mathbf{1}),$$

where the last sum is of exponential decay when $v = \mathrm{Im}(\tau) \to \infty$. This proves the proposition. ■

5 The main formula

Let L be an even integral lattice in V, and let $K \subset G(\hat{\mathbb{Q}})$ be a compact open subgroup which fixes L and acts trivially on L'/L. We also assume that K satisfies the condition

$$K \cap \mathbb{G}_m(\hat{\mathbb{Q}}) = \hat{\mathbb{Z}}^\times, \tag{5.1}$$

where $\mathbb{G}_m$ is the kernel of the homomorphism $\mathrm{GSpin}(V) \to \mathrm{SO}(V)$. Let $f \in H_{1-d,\bar{\rho}_L}$ be a harmonic weak Maass form and let $\Phi(z, h, f)$ be the corresponding 'automorphic' Green function for the divisor $Z(f)$ defined in (3.14).

For $\vec{\tau} \in \mathbb{H}^{d+1}$ and $\tau \in \mathbb{H}$, define $S_L^\vee$-valued functions by

$$E(\vec{\tau}, s, L, \mathbf{1}) = \sum_{\mu \in L'/L} E(\vec{\tau}, s, \varphi_\mu, \mathbf{1})\, \varphi_\mu^\vee, \qquad \mathcal{E}(\tau, L) = \sum_{\mu \in L'/L} \mathcal{E}(\tau, \varphi_\mu)\, \varphi_\mu^\vee, \tag{5.2}$$

where $\mathcal{E}(\tau, \varphi)$ is defined in Proposition 4.6, and the normalized incoherent Eisenstein series

$$E^*(\vec{\tau}, s, L, \mathbf{1}) = \Lambda(s+1, \chi)\, E(\vec{\tau}, s, L, \mathbf{1}).$$

Define the L-function for a cuspidal modular form $g = \sum_\mu g_\mu \varphi_\mu \in S_{d+1,\rho_L}$

$$\mathcal{L}(s, g, L) = \langle E^*(\tau^\Delta, s, L, \mathbf{1}), g\rangle_{\mathrm{Pet}} := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \sum_\mu \overline{g_\mu(\tau)}\, E^*(\tau^\Delta, s, \varphi_\mu, \mathbf{1})\, v^{d+1}\, d\mu(\tau). \tag{5.3}$$

It can be viewed as the g-isotypical component of diagonal restriction of the Hilbert-Eisenstein series $E(\vec{\tau}, s, L, \mathbf{1})$.

Remark 5.1. The Eisenstein series $E(\vec{\tau}, s, L, \mathbf{1})$ depends on the F-quadratic form on $L \otimes \mathbb{Q} = W$, not just on the $\mathbb{Q}$-quadratic form on $L \otimes \mathbb{Q} = V$. When we need to emphasize this dependence on the F-quadratic form, we will write $L(W)$ rather than L and

$$E^*(\vec{\tau}, s, L(W), \mathbf{1}) = E^*(\vec{\tau}, s, L, \mathbf{1}), \quad \mathcal{E}(\tau, L(W)) = \mathcal{E}(\tau, L), \quad \mathcal{L}(s, g, L(W)) = \mathcal{L}(s, g, L)$$

We also caution that $L(W)$ might not be an O_F-lattice, i.e., it might not be O_F-invariant. □

Since the Eisenstein series has an analytic continuation and is incoherent, the L-series $\mathcal{L}(s, g, L)$ has an analytic continuation and is zero at the central point $s = 0$. Now we are ready to state and prove the main formula. Here, if $\sum_n a_n q^n$ is a power series in q, we write

$$\mathrm{CT}\,\Big[\sum_n a_n q^n\Big] = a_0$$

for the constant term.

Theorem 5.2. For a harmonic weak Maass form $f \in H_{1-d,\bar{\rho}_L}$ with components $f = f^+ + f^-$ as in (3.7) and with other notation as above,

$$\Phi(Z(W), f) = C(W, K)\Big(\ \mathrm{CT}\,\big[\langle f^+(\tau), \mathcal{E}(\tau, L(W))\rangle\big] - \mathcal{L}'(0, \xi(f), L(W))\ \Big),$$

where $\xi(f)$ is the image of f under the anti-holomorphic operator $\xi : H_{1-d,\bar{\rho}_L} \to S_{d+1,\rho_L}$, cf. (3.8), and

$$C(W, K) = \frac{\deg(Z(T, z_0^{\pm}))}{\Lambda(0, \chi)}.$$

□

Proof. The proof basically follows the same argument as in [10, Theorem 4.8]. We write L in place of $L(W)$. First, by Lemma 4.3 and Proposition 4.5, we have

$$\begin{aligned}\Phi(Z(T(j), z_0(j), g_j), f) &= \int_{\mathcal{F}}^{\mathrm{reg}} \langle f(\tau), \theta_L(\tau, Z(T(j), z_0(j), g_j))\rangle\, d\mu(\tau)\\ &= C \int_{\mathcal{F}}^{\mathrm{reg}} \langle f(\tau), E(\tau^{\Delta}, 0, L, \mathbf{1}(j))\, d\mu(\tau)\rangle\\ &= -2C \int_{\mathcal{F}}^{\mathrm{reg}} \langle f(\tau), \bar{\partial}_j(E'(\tau^{\Delta}, 0, L, \mathbf{1})\, d\tau)\rangle.\end{aligned}$$

Here C is the constant in Proposition 4.5. So, summing on j, and recalling the definition (2.13) of $Z(W)$, we have

$$\begin{aligned}\Phi(Z(W),f) &= -4\,C\int_{\mathcal{F}}^{\mathrm{reg}} \langle f(\tau), \sum_j \bar\partial_j(E'(\tau^\Delta,0,L,\mathbf{1})\,d\tau)\rangle \\ &= -4C\int_{\mathcal{F}}^{\mathrm{reg}} \langle f(\tau), \bar\partial(E'(\tau^\Delta,0,L,\mathbf{1})\,d\tau)\rangle \\ &= -4C\int_{\mathcal{F}}^{\mathrm{reg}} d(\langle f(\tau), E'(\tau^\Delta,0,L,\mathbf{1})\,d\tau\rangle) + 4C\int_{\mathcal{F}}^{\mathrm{reg}} \langle \bar\partial f(\tau), E'(\tau^\Delta,0,L,\mathbf{1})\,d\tau\rangle \\ &= -C_0 I_1 + C_0 I_2,\end{aligned}$$

where $C_0 = 4C\Lambda(0,\chi)^{-1} = C(W,K)$, and

$$\begin{aligned} I_1 &= \int_{\mathcal{F}}^{\mathrm{reg}} d(\langle f(\tau), E^{*,\prime}(\tau^\Delta,0,L,\mathbf{1})\,d\tau\rangle), \\ I_2 &= \int_{\mathcal{F}}^{\mathrm{reg}} \langle \bar\partial f(\tau), E^{*,\prime}(\tau^\Delta,0,L,\mathbf{1})\,d\tau\rangle.\end{aligned}$$

Recall that

$$\bar\partial f(\tau) = -\frac{1}{2i} v^{d-1}\overline{\xi(f)}\,d\bar\tau.$$

Thus

$$\langle \bar\partial f(\tau), E^{*,\prime}(\tau^\Delta,0,L,\mathbf{1})\,d\tau\rangle = -\langle \overline{\xi(f)}, E^{*,\prime}(\tau^\Delta,0,L,\mathbf{1})\rangle\, v^{d+1}\,d\mu(\tau)$$

is actually integrable over the fundamental domain $\mathcal{F}$, and hence

$$I_2 = -\int_{\mathcal{F}} \langle \overline{\xi(f)}, E^{*,\prime}(\tau^\Delta,0,L,\mathbf{1})\rangle\, v^{d+1} d\mu(\tau) = -\mathcal{L}'(0,\xi(f),L).$$

By the same argument as in [20, Proposition 2.5], [37, Proposition 2.19], or [10, Lemma 4.6], there is a (unique) constant A_0 such that

$$I_1 = \lim_{T\to\infty}\left(\int_{\mathcal{F}_T} d(\langle f(\tau), E^{*,\prime}(\tau^\Delta,0,L,\mathbf{1})\,d\tau\rangle) - A_0\log T\right) = \lim_{T\to\infty}(I_1(T) - A_0\log T).$$

By Stokes' theorem, one has

$$\begin{aligned} I_1(T) &= \int_{\partial\mathcal{F}_T} \langle f(\tau), E^{*,\prime}(\tau^\Delta,0,L,\mathbf{1})\rangle\,d\tau \\ &= -\int_{iT}^{iT+1} \langle f(\tau), E^{*,\prime}(\tau^\Delta,0,L,\mathbf{1})\rangle\,du \\ &= -\int_{iT}^{iT+1} \langle f^+(\tau), E^{*,\prime}(\tau^\Delta,0,L,\mathbf{1})\rangle\,du + O(e^{-\epsilon T})\end{aligned}$$

for some $\epsilon > 0$ since f^- is of exponential decay and $E^{*,\prime}$ is of moderate growth. Proposition 4.6 asserts that

$$E^{*,\prime}(\tau^\Delta, 0, L) = \mathcal{E}(\tau, L) + \Lambda(0,\chi)\,(d+1)\,\log(v) + \sum_{\mu\in L'/L}\sum_{m\in\mathbb{Q}} a(m,\mu,v)\,q^m$$

such that $a(m,\mu,v)\,q^m$ is of exponential decay as $v\to\infty$. Thus,

$$-I_1(T) = \mathrm{CT}[\langle f^+(\tau), \mathcal{E}(\tau,L)\rangle] + \Lambda(0,\chi)\,(d+1)\log T + \sum_{\mu\in L'/L}\sum_{m+n=0} c^+(m,\mu)a(n,\mu,T).$$

The last sum goes to zero when $T\to\infty$. So we can take $A_0 = (d+1)\,\Lambda(0,\chi)$, and

$$I_1 = -\,\mathrm{CT}[\langle f^+(\tau), \mathcal{E}(\tau,L)\rangle]$$

as claimed. ■

Remark 5.3. There is a sign error in front of $\mathcal{L}'(\xi(f), U, 0)$ in [10, Theorem 4.7] and throughout that paper caused by this error. The $+\mathcal{L}'(\xi(f), U, 0)$ in that theorem should be $-\mathcal{L}'(\xi(f), U, 0)$. Accidently, in the proof of [10, Theorem 7.7], there is another sign error relating the Faltings' height and the Neron-Tate height. Two wrong signs give the correct formula in [10, Theorem 7.7], which somehow prevented the authors from discovering the sign error earlier. □

As in [10], this theorem raises two interesting conjectures. We very briefly describe them and refer to [10, Section 5] for details. Assume that there is a regular scheme $\mathcal{X}_K \to \operatorname{Spec}\mathbb{Z}$, projective and flat over $\mathbb{Z}$, whose associated complex variety is a smooth compactification X_K^c of X_K. Let $\mathcal{Z}(m,\mu)$ and $\mathcal{Z}(W)$ be suitable extensions to $\mathcal{X}_K$ of the cycles $Z(m,\mu)$ and $Z(W)$, respectively. Such extensions can be found in low dimensional cases using a moduli interpretation of $\mathcal{X}_K$. For an $f\in H_{1-g,\bar\rho_L}$, the function $\Phi(f)$ is a Green function for the divisor $Z(f)$. Set $\mathcal{Z}(f) = \sum_\mu\sum_{m>0} c^+(-m,\mu)\mathcal{Z}(m,\mu)$. Then the pair

$$\hat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(\cdot, f))$$

defines an arithmetic divisor in $\widehat{\mathrm{CH}}^1(\mathcal{X}_K)_{\mathbb{C}}$. Theorem 5.2 provides a formula for the quantity

$$\langle \hat{\mathcal{Z}}(f), \mathcal{Z}(W)\rangle_\infty = \frac{1}{2}\Phi(Z(W), f), \tag{5.4}$$

and inspires the following 'equivalent' conjectures.

Conjecture 5.4. *Let $\mu\in L'/L$, and let $m\in Q(\mu)+\mathbb{Z}$ be positive. Then $\mathcal{Z}(m,\mu)$ and $\mathcal{Z}(W)$ intersect properly, and the arithmetic intersection number $\langle\mathcal{Z}(m,\mu),\mathcal{Z}(W)\rangle_{fin}$ is equal to $-\frac{1}{2}C(W,K)$ times the (m,μ)-th Fourier coefficient of $\mathcal{E}(\tau,L)$.* □

Conjecture 5.5. *For any $f \in H_{1-d,\bar{\rho}_L}$, one has*

$$\langle \hat{\mathcal{Z}}(f), \mathcal{Z}(W) \rangle_{Fal} = \frac{1}{2} C(W,K) \left(c^+(0,0)\kappa(0,0) - \mathcal{L}'(0, \xi(f), L) \right). \tag{5.5}$$

Here $\kappa(0,0)$ is the constant term of $\mathcal{E}(\tau, L)$ □

6 Hilbert modular surfaces

In general, the Shimura varieties attached to orthogonal groups of signature $(n, 2)$ are of Hodge but not PEL type, so our special cycles do not have a simple description in terms of moduli of abelian varieties. However, in the case of signature $(2, 2)$, such an interpretation is always possible, cf. [23]. In this section, we consider the case of Hilbert modular surfaces and explain how our earlier construction of 0-cycles can be given a modular interpretation. The most efficient way to do this is based on the machinery set up in [17], where the quadratic space of signature $(2, 2)$ arises as a space of special endomorphisms and the quartic CM points are linked to the cycles of section 2 by a reflex field construction.

6.1 RM abelian surfaces

Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field with fundamental discriminant D, ring of integers O_F, and different $\partial = \partial_F = \sqrt{D} O_F$. We fix an embedding of F into $\mathbb{R}$ with $\sqrt{D} > 0$ and write σ for the other real embedding and for the nontrivial Galois automorphism of $F/\mathbb{Q}$.

We begin with some algebra. Let M be a free $\mathbb{Z}$-module of rank 4 with a perfect alternating pairing $\lambda : M \times M \to \mathbb{Z}$. For an element $j \in \mathrm{End}(M)$, let j^* be the adjoint of j with respect to λ, i.e., $\lambda(jx, y) = \lambda(x, j^* y)$. Let $\kappa : O_F \hookrightarrow \mathrm{End}(M)$ be an action of O_F on M such that $\kappa(a)^* = \kappa(a)$. Such a triple (M, κ, λ) is unique up to isomorphism and is called a principally polarized RM module in [17]. Note that there is a unique O_F-bilinear alternating pairing $\Lambda : M \times M \to \partial^{-1}$ such that $\lambda(x, y) = \mathrm{tr}_{F/\mathbb{Q}}(\Lambda(x, y))$. Define the lattice $L(M)$ of special endomorphisms of (M, κ, λ) by

$$L(M) = \{\ j \in \mathrm{End}(M) : j \circ \kappa(a) = \kappa(\sigma(a)) \circ j,\ j^* = j\ \},$$

and let $V(M) = L(M) \otimes_{\mathbb{Z}} \mathbb{Q}$. There is a $\mathbb{Z}$-valued quadratic form Q on $L(M)$ defined by $j^2 = Q(j)\, 1_M$.

As a standard principally polarized RM module $(\overset{o}{M}, \overset{o}{\kappa}, \overset{o}{\lambda})$, we take $\overset{o}{M}_{\mathbb{Q}} = F^2$, (column vectors),

$$\overset{o}{M} = \{x\ : {}^t x \in O_F \oplus \partial^{-1}\}$$

and $\overset{o}{\Lambda}(x,y) = x_1y_2 - x_2y_1$, where ${}^tx = (x_1,x_2)$ and ${}^ty = (y_1,y_2)$. Then

$$\mathrm{End}_{O_F}(\overset{o}{M}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,\ d \in O_F,\ c \in \partial^{-1},\ b \in \partial \}.$$

Let $\Gamma = \mathrm{End}_{O_F}(\overset{o}{M}) \cap \mathrm{SL}_2(F)$. Let G be the algebraic group over $\mathbb{Q}$ defined by

$$G(R) = \{ g \in \mathrm{GL}_2(F \otimes_{\mathbb{Q}} R) : \det(g) \in R^\times \},$$

and note that $\Gamma \subset G(\mathbb{Q})$ is an arithmetic subgroup. There is a natural homomorphism $G \to \mathrm{GSp}(\overset{o}{M}_{\mathbb{Q}}, \overset{o}{\lambda})$ whose image is precisely the O_F-linear similitudes, and the restriction of the similitude scale to G is given by the determinant. Let Z be the center of G and note that the map $r \mapsto r \cdot 1_2$ gives an isomorphism of $\mathbb{R}^\times$ with a subgroup of $Z(\mathbb{R})$ of index 2.

Define $j_0 \in L(\overset{o}{M})$ by $j_0({}^t(x_1,x_2)) = {}^t(\sigma(x_1),\sigma(x_2))$. Then there is an isomorphism

$$\{ \begin{pmatrix} a & b\sqrt{D} \\ c/\sqrt{D} & \sigma(a) \end{pmatrix} : a \in O_F,\ b,c \in \mathbb{Z} \} \overset{\sim}{\longrightarrow} L(\overset{o}{M}), \qquad x \mapsto x \circ j_0.$$

Under this isomorphism, the quadratic form Q on $L(\overset{o}{M})$ is given by $Q(x \circ j_0) = \det(x) = N_{F/\mathbb{Q}}(a) - bc$, and, in particular, has signature $(2,2)$. Note that the natural action of $G(\mathbb{Q})$ on $V := V(\overset{o}{M})$ by conjugation preserves the quadratic form Q, and this provides an identification $G \overset{\sim}{\longrightarrow} \mathrm{GSpin}(V,Q)$. For any principally polarized RM module (M,κ,λ), we can choose an isomorphism

$$(*): \qquad (M,\kappa,\lambda) \overset{\sim}{\longrightarrow} (\overset{o}{M},\overset{o}{\kappa},\overset{o}{\lambda})$$

with the standard module. Any two such isomorphisms differ by an element of Γ.

Let

$$\mathcal{D} = \{ \mathtt{h} \in G(\mathbb{R}) : \mathtt{h}^2 = -1,\ \overset{o}{\lambda}(\mathtt{h}x,y) \text{ is symmetric and positive definite} \}$$

be the space of admissible complex structures on $\overset{o}{M}_{\mathbb{R}} = \overset{o}{M} \otimes_{\mathbb{Z}} \mathbb{R}$. Such an $\mathtt{h}$ determines a homomorphism $\underline{\mathtt{h}} : \mathbb{S} \to G_{\mathbb{R}}$ given on real points by $\mathbb{C}^\times \to G(\mathbb{R})$, $\underline{\mathtt{h}}(a+ib) = a + \mathtt{h}b$, where we identify $\mathbb{R}^\times$ with a subgroup of $Z(\mathbb{R})$, as above. Let $\mathbb{D}$ be the space of oriented negative 2-planes in $V(\mathbb{R})$, where $V = V(\overset{o}{M})$. As explained in [17, section 3.4], there is an isomorphism $\mathcal{D} \overset{\sim}{\longrightarrow} \mathbb{D}^+ \subset \mathbb{D}$ of $\mathcal{D}$ with one of the two components of $\mathbb{D}$ given by sending $\mathtt{h} \in \mathcal{D}$ to

$$z(\mathtt{h}) = \{ j \in V(\mathbb{R}) : j \circ \mathtt{h} = -\mathtt{h} \circ j \}.$$

The orientation of $z(\mathtt{h})$ is given by the action of $\underline{\mathtt{h}}(\mathbb{C}^\times)$ on it by conjugation. If an oriented negative 2-plane $z \in \mathbb{D}^+$ is given, we write $\mathtt{h}_z$ for the corresponding complex structure on $\overset{o}{M}_\mathbb{R}$.

Recall that a principally polarized RM abelian surface over a connected scheme S is a collection (A, κ_A, λ_A) consisting of an abelian scheme A over S together with an action $\kappa_A : O_F \to \mathrm{End}(A)$ and a principal polarization $\lambda_A : A \to A^\vee$ such that $\kappa_A(a)^* = \kappa_A(a)$ for all $a \in O_F$, where $*$ denotes the Rosati involution associated to λ_A. Moreover, the Kottwitz condition is imposed, i.e., for $a \in O_F$, the characteristic polynomial of the endomorphism of Lie(A) determined by $\kappa_A(a)$ is required to coincide with the polynomial $(T-a)(T-\sigma(a))$. Let $\mathcal{M}$ be the moduli stack over Spec$(\mathbb{Z})$ of principally polarized RM surfaces – cf [17, section 3.1] for more details.

If a principally polarized RM abelian surface (A, κ, λ) over $\mathbb{C}$ is given, $(H_1(A, \mathbb{Z}), \kappa, \lambda)$ is a principally polarized RM module with admissible complex structure determined by the isomorphism $H_1(A, \mathbb{Z}) \otimes \mathbb{R} = \mathrm{Lie}(A)$. If we choose an isomorphism (*) of $(H_1(A, \mathbb{Z}), \kappa, \lambda)$ with $(\overset{o}{M}, \overset{o}{\kappa}, \overset{o}{\lambda})$, this complex structure determines a point of $\mathcal{D} \simeq \mathbb{D}^+$. The action of Γ removes the ambiguity involved in the choice of (*). Conversely, for $z \in \mathbb{D}^+$, there is a principally polarized RM surface A_z over $\mathbb{C}$ determined by the data $(\overset{o}{M}_\mathbb{R}, \overset{o}{M}, \kappa, \overset{o}{\lambda}, \mathtt{h}_z)$. The action of an element $\gamma \in \Gamma$ on $\overset{o}{M}$ yields an O_F-linear isomorphism of A_z and $A_{\gamma(z)}$. In this way, we obtain a uniformization of the space of RM abelian surfaces over $\mathbb{C}$. Indeed, the map $z \mapsto A_z$ induces an isomorphism of orbifolds

$$[\Gamma \backslash \mathcal{D}] \simeq [\Gamma \backslash \mathbb{D}^+] \overset{\sim}{\longrightarrow} \mathcal{M}(\mathbb{C}).$$

The image of a point $z \in \mathbb{D}^+$ is counted with multiplicity $|\Gamma_z|^{-1}$. Finally, if we let

$$K = \mathrm{Aut}_{O_F}(\overset{o}{M} \otimes \widehat{\mathbb{Z}}) \cap G(\widehat{\mathbb{Q}}),$$

and let $G(\mathbb{R})^+$ be the stabilizer of $\mathbb{D}^+$ in $G(\mathbb{R})$, we have $\Gamma = G(\mathbb{Q}) \cap G(\mathbb{R})^+ K$, and, by strong approximation,

$$X_K(\mathbb{C}) = [G(\mathbb{Q}) \backslash (\mathbb{D} \times G(\widehat{\mathbb{Q}})/K)] \simeq [\Gamma \backslash \mathbb{D}^+] \overset{\sim}{\longrightarrow} \mathcal{M}(\mathbb{C}),$$

where $X_K(\mathbb{C})$ is the orbifold of complex points of the Shimura variety X_K arising from V as in section 2.

6.2 Non-biquadratic CM points

Next we turn to the quartic CM points, again beginning with some algebra. Let E be a non-biquadratic quartic CM field with real subfield F and denote the non-trivial Galois automorphism of E/F by $a \mapsto \bar{a}$. Again following [17], we consider principally polarized CM modules (M, κ_E, λ) where M and λ are as before and $\kappa_E : O_E \hookrightarrow \mathrm{End}(M)$ is an O_E-action with $\kappa_E(a)^* = \kappa_E(\bar{a})$. In particular, M is a projective O_E-module of rank 1, and $(M, \kappa_E|_{O_F}, \lambda)$ is a principally polarized RM module. Let $V(M)$ be the space of special endomorphisms of $(M, \kappa_E|_{O_F}, \lambda)$. We now recall the definition of the reflex action on $V(M)$ introduced in [17, section 2]. Consider the $\mathbb{Q}$-algebra

$$R = E \otimes_{id,F,\sigma} E,$$

and define automorphisms τ and ρ by $\tau(a \otimes b) = b \otimes a$ and $\rho(a \otimes b) = \bar{b} \otimes a$. Then $E^\sharp = R^{\tau=1}$, the subalgebra fixed by τ, is again a non-biquadratic CM field with 'real' subfield $F^\sharp = R^{\tau=1,\rho^2=1}$. Let $a \mapsto a^\dagger = \rho^2(a)$ denote the 'complex conjugation' $E^\sharp$. (The quotation marks are due to the fact that $E^\sharp$ is not identified with a subfield of $\mathbb{C}$.) Note that $E^\sharp$ is spanned by elements of the form $\alpha = a \otimes b + b \otimes a$, for a and $b \in E$, and that there is a norm map $N^\sharp : E \longrightarrow E^\sharp$, $a \mapsto a \otimes a$. A key observation from [17] is that the action κ_E of O_E on M determines an action of $E^\sharp$ on $V(M)$ given by

$$\alpha \bullet j = (a \otimes b + b \otimes a) \bullet j = \kappa_E(a) \circ j \circ \kappa_E(\bar{b}) + \kappa_E(b) \circ j \circ \kappa_E(\bar{a}).$$

Moreover, if $(\ ,\ )$ is the bilinear form on $V(M)$ associated to Q, then $(\alpha \bullet j_1, j_2) = (j_1, \alpha^\dagger \bullet j_2)$, where $\alpha^\dagger = \rho^2(\alpha)$ is the 'complex conjugation' on $E^\sharp$. It follows that there is a unique hermitian form $(\ ,\ )_{E^\sharp}$ on the 1-dimensional $E^\sharp$ vector space $V(M)$ such that $(x, y) = \mathrm{tr}_{E^\sharp/\mathbb{Q}}(x, y)_{E^\sharp}$. Similarly, we can view $V(M)$ as a 2-dimensional vector space over $F^\sharp$ with a unique $F^\sharp$-quadratic form $Q^\sharp_M$ such that $Q(x) = \mathrm{tr}_{F^\sharp/\mathbb{Q}} Q^\sharp_M(x)$. We will write $(W(M), Q^\sharp_M)$ for this space, although it depends, of course, on the principally polarized CM module (M, κ_E, λ). Notice that $Q^\sharp_{\mathbb{M}}$ depends on the CM module (M, κ_E, λ) although Q does not.

It is easily checked, cf. [17, Section 2], that the four CM types of the field E are in bijection with the four complex embeddings of the field $E^\sharp$, via the map $\Sigma \mapsto \phi_\Sigma$ given by

$$\phi_\Sigma(\alpha) = \phi_\Sigma(a \otimes b + b \otimes a) = \tau_1(a)\tau_2(b) + \tau_1(b)\tau_2(a), \qquad \Sigma = \{\tau_1, \tau_2\},$$

where the τ_i, $1 \le i \le 4$, are the complex embeddings of E. A CM type Σ of E determines an isomorphism $i_\Sigma : E_\mathbb{R} = E \otimes_\mathbb{Q} \mathbb{R} \xrightarrow{\sim} \mathbb{C}^2$ and hence, if a principally polarized CM module (M, κ_E, λ) is given, Σ determines a complex structure $\mathtt{h}_\Sigma$ on $M_\mathbb{R}$, via the action of the diagonally embedded $\mathbb{C}$ in $\mathbb{C}^2$, i.e,

$$\mathtt{h}_\Sigma = \kappa_E \circ i_\Sigma^{-1}(i, i). \tag{6.1}$$

Moreover, for a given principally polarized CM module (M, κ_E, λ), there exists a unique CM type Σ of E such that the form $\lambda(\mathtt{h}_\Sigma x, y)$ is symmetric and positive definite, [17, Section 2.3], in which case, (M, κ_E, λ) is said to be of type Σ. The following result is [17, Proposition 2.3.5].

Lemma 6.1. Suppose that (M, κ_E, λ) is of type Σ. Let ∞^-_Σ be the restriction of ϕ_Σ to $F^\sharp$ and let ∞^+_Σ be the other real embedding of $F^\sharp$. Then the signature of the binary quadratic space $(W(M), Q^\sharp_M)$ over $F^\sharp$ is $(0,2)$ at ∞^-_Σ and $(2,0)$ at ∞^+_Σ. □

For a fixed CM type $\Sigma = \{\tau_1, \tau_2\}$ of E, let O_Σ be the ring of integers in $E_\Sigma = \phi_\Sigma(E^\sharp)$. For a connected

O_Σ-scheme S, a principally polarized CM abelian surface over S is a collection (A, κ_A, λ_A) consisting of an abelian scheme A over S with an action $\kappa_A : O_E \to \mathrm{End}(A)$ of O_E and a principal polarization λ_A such that, for $b \in O_E$, $\kappa_A(b)^* = \kappa_A(\bar{b})$. In addition, the Σ-Kottwitz condition is imposed, i.e., the characteristic polynomial of the endomorphism of $\mathrm{Lie}(A)$ determined by $\kappa_A(b)$ coincides with the the image of $(T - \tau_1(b))(T - \tau_2(b)) \in O_\Sigma[T]$ in $\mathcal{O}_S[T]$. We let $\mathcal{CM}_E^\Sigma$ be the moduli stack over $\mathrm{Spec}(O_\Sigma)$ of CM abelian surfaces of type Σ – cf. [17, Section 3.3] for more details.

It will be useful to consider the action of an ideal class group on the CM points. Let $C(E)$ be the generalized ideal class group of E whose elements are equivalence classes $[\mathfrak{a}, \xi]$ of pairs $(\mathfrak{a}, \xi)$, where $\mathfrak{a}$ is a fractional ideal of E and $\xi \in F^\times$ is such that $N_{E/F}\mathfrak{a} = \xi\, O_F$. Pairs $(\mathfrak{a}_1, \xi_1)$ and $(\mathfrak{a}_2, \xi_2)$ are equivalent if there is an element $\alpha \in E^\times$ such that $\mathfrak{a}_1 = \alpha\mathfrak{a}_2$ and $\xi_1 = N_{E/F}(\alpha)\, \xi_2$. Let $C_+(E)$ be the subgroup of classes $[\mathfrak{a}, \xi]$ for which ξ is totally positive. For a pair $\mathbf{c} = (\mathfrak{a}, \xi)$ with ξ totally positive and a principally polarized CM abelian surface (A, κ_A, λ_A) define $(B, \kappa_B, \lambda_B) = \mathbf{c} \bullet (A, \kappa_A, \lambda_A)$ as follows. Let $B = \mathfrak{a} \otimes_{O_E} A$ be given by the Serre construction, [33], and let $\kappa_B(r) = r \otimes 1_A$ be the natural action. Let $f \in \mathrm{Hom}(B, A) \otimes_\mathbb{Z} \mathbb{Q}$ be the natural O_E-linear quasi-isogeny defined by $f(a \otimes x) = \kappa_A(a)x$, and define the polarization λ_B by

$$\lambda_B = f^\vee \circ \lambda_A \circ f \circ \kappa_B(\xi^{-1}).$$

As explained in [17, Section 3, pp. 29–30], (B, κ_B, λ_B) is again a principally polarized CM abelian surface, and this construction defines an action of such pairs on the stack $\mathcal{CM}_E^\Sigma$.

Now suppose that (A, κ_A, λ_A) is in $\mathcal{CM}_E^\Sigma(\mathbb{C})$. The associated CM modules $(M(A), \kappa_A, \lambda_A)$ and $(M(B), \kappa_B, \lambda_B)$, where $M(A) = H_1(A, \mathbb{Z})$ and $M(B) = H_1(A, \mathbb{Z})$, are related by $M(B) = \mathfrak{a} \otimes_{O_E} M(A)$ and the polarization form λ_B on $M(B)$ is given by

$$\lambda_B(a \otimes x, b \otimes y) = \lambda_A(\kappa_A(\xi^{-1}\, a\bar{b})x, y). \tag{6.2}$$

In particular, there is a natural E-linear isomorphism

$$f_* : M(B)_\mathbb{Q} = \mathfrak{a} \otimes_{O_E} M(A)_\mathbb{Q} \xrightarrow{\sim} M(A)_\mathbb{Q}, \qquad a \otimes x \mapsto \kappa_A(a)x. \tag{6.3}$$

If we identify $M(B)_\mathbb{Q}$ and $M(A)_\mathbb{Q}$ via f_* and if $t \in \widehat{E}^\times$ is such that $\mathfrak{a} = tO_E$, then the lattice $M(B)$ is identified with the lattice $\kappa_A(t)M(A)$. The isomorphism f_* also induces an $E^\sharp$-linear isomorphism of the spaces of special endomorphisms

$$V(M(B)) \xrightarrow{\sim} V(M(A)), \qquad j \mapsto \kappa_A(\xi^{-1}) \cdot f_* \circ j \circ f_*^{-1}. \tag{6.4}$$

which gives an isometry of the $F^\sharp$-quadratic spaces $(W(M(B)), Q^\sharp_{M(B)})$ and $(W(M(A)), N_{F/\mathbb{Q}}(\xi)\, Q^\sharp_{M(A)})$. Note that there is an isometry of rational quadratic spaces $(V(M(A)), N_{F/\mathbb{Q}}(\xi)\, Q) \simeq (V(M(A)), Q)$, but this map is not $F^\sharp$-linear, in general. The principally polarized CM modules $(M(A), \kappa_A, \lambda_A)$ and $(M(B), \kappa_B, \lambda_B)$ have the same CM type.

In fact, the action of pairs $\mathbf{c} = (\mathfrak{a}, \xi)$ on CM modules is well defined with no positivity restriction on ξ, where the CM type Σ is changed according to the signature of ξ. For example, the pair $(O_F, -1)$ carries (M, κ_E, λ), of type Σ, to $(M, \kappa_E, -\lambda)$, of type $\bar{\Sigma}$. The results of [17, Sections 2 and 3] imply the following.

Proposition 6.2. Fix a principally polarized abelian surface $\mathbf{A} = (A, \kappa_A, \lambda_A)$ over $\mathbb{C}$ of type Σ.

(i) There is an isomorphism of orbifolds

$$[\mu(E)\backslash C_+(E)] \xrightarrow{\sim} \mathcal{CM}_E^\Sigma(\mathbb{C}), \qquad \mathbf{c} \mapsto \mathbf{c} \bullet \mathbf{A},$$

where $\mu(E)$ is the group of roots of unity in E, and this group acts trivially on $C_+(E)$.

(ii) Suppose that $\mathbf{c} = (\mathfrak{a}, \xi)$ is a pair representing a class $[\mathfrak{a}, \xi] \in C(E)$. Choose $t \in \widehat{E}^\times$ such that $\mathfrak{a} = tO_E$, and let $\mathrm{gen}_f(\mathbf{c}) = \xi^{-1}\, t\, \bar{t} \in \widehat{O}_F^\times / N_{E/F}(\widehat{O}_E^\times)$ be the genus invariant of $\mathbf{c}$. For a principally polarized CM module $\mathbf{M} = (M, \kappa, \lambda)$ of type Σ, let $\mathbf{M}' = (M', \kappa', \lambda') = \mathbf{c} \bullet \mathbf{M}$. Then there is an isometry

$$(\widehat{W}(M'), Q^\sharp_{M'}) \xrightarrow{\sim} (\widehat{W}(M), N_{F/\mathbb{Q}}(\widetilde{\mathrm{gen}}_f(\mathbf{c}))^{-1} \cdot Q^\sharp_M)$$

of quadratic spaces over $\widehat{F}^\sharp$ carrying $\widehat{L}(M')$ to $\widehat{L}(M)$. Here $\widetilde{\mathrm{gen}}_f(\mathbf{c}) \in \widehat{O}_F^\times$ is any representative for $\mathrm{gen}_f(\mathbf{c})$.

□

Remark 6.3. By [17, (2.4.3)], if

$$\mathrm{Gen}(E/F) = F_\infty^\times / N_{E/F}(E_\infty^\times) \times \widehat{O}_F^\times / N_{E/F}(\widehat{O}_E^\times),$$

and it $\chi_{E/F}$ is the quadratic character of $F_\infty^\times \times \widehat{F}^\times$ associated to E/F, then

$$C(E) \xrightarrow{\mathrm{gen}} \mathrm{Gen}(E/F) \xrightarrow{\chi_{E/F}} \{\pm 1\} \longrightarrow 1,$$

is exact.

□

Slightly smaller orbits will also be of interest. Let T_E and $S_{E^\sharp}$ be algebraic tori over $\mathbb{Q}$, where, for any $\mathbb{Q}$-algebra R,

$$T_E(R) = \{\ t \in (E \otimes_{\mathbb{Q}} R)^\times : t\bar{t} \in R^\times\ \}, \qquad \text{and} \qquad S_{E^\sharp}(R) = \{\ t \in (E^\sharp \otimes_{\mathbb{Q}} R)^\times : t\bar{t} = 1\ \}.$$

Note that $S_{E^\sharp} = \mathrm{Res}_{F^\sharp/\mathbb{Q}}\, \mathrm{SO}(W(M)) \subset \mathrm{SO}(V(M))$, while, the action of $T_E(\mathbb{Q})$ on $V(M)$ defined by

$$t \bullet j = \frac{1}{t\bar{t}}\, \kappa_E(t) \circ j \circ \kappa_E(\bar{t})$$

lifts to an embedding $T_E \subset \mathrm{GSpin}(V(M))$ compatible with the exact sequence, [17, Lemma 1.4.1],

$$1 \longrightarrow \mathbb{G}_m \longrightarrow T_E \xrightarrow{\nu_E} S_{E^\sharp} \longrightarrow 1, \qquad \nu_E(t) = \frac{N^\sharp(t)}{t\bar{t}}. \tag{6.5}$$

Let $K_T = T_E(\widehat{\mathbb{Q}}) \cap \widehat{O}_E^\times$. Then there is an injective homomorphism, [17, (2.4.1)],

$$C(T_E) := T_E(\mathbb{Q})\backslash T_E(\widehat{\mathbb{Q}})/K_T \longrightarrow C_+(E), \qquad [t] \mapsto [\mathbf{t}], \quad \mathbf{t} := (tO_E, \xi),$$

where tO_E is the fractional ideal generated by t and $\xi \in \mathbb{Q}_{>0}^\times$ is the unique element such that $t\bar{t}\mathbb{Z} = \xi\mathbb{Z}$.

For a fixed principally polarized CM abelian surface $\mathbf{A} = (A, \kappa_A, \lambda_A)$ over $\mathbb{C}$ of type Σ, as above, we obtain a variant of Proposition 6.2. There is an isomorphism of orbifolds

$$[T_E(\mathbb{Q})\backslash(T_E(\widehat{\mathbb{Q}})/K_T)] = [\mu(E)\backslash C(T_E)] \xrightarrow{\sim} Z(C(T_E), \mathbf{A}) \subset \mathcal{CM}_E^\Sigma(\mathbb{C}), \tag{6.6}$$

where $Z(C(T_E), \mathbf{A})$ is the subgroupoid whose objects are the $\mathbf{t} \bullet \mathbf{A}$ for $t \in T_E(\widehat{\mathbb{Q}})$.

Finally, there is an action of the group $\mathrm{Aut}(\mathbb{C}/E_\Sigma)$ on $\mathcal{CM}_E^\Sigma(\mathbb{C})$ described by the theory of complex multiplication, [34], [36], [26], [28]. We adopt (some of) the notation of Milne. Let E_Σ be the reflex field of (E, Σ) and let $N_\Sigma : E_\Sigma^\times \longrightarrow E^\times$ be the reflex norm map. If $\eta \in \mathrm{Aut}(\mathbb{C}/E_\Sigma)$ and $s \in \widehat{E}_\Sigma^\times$ is such that $\eta\,|_{E_\Sigma^{\mathrm{ab}}} = \mathrm{art}_{E_\Sigma}(s)$, let $t = N_\Sigma(s) \in T_E(\widehat{\mathbb{Q}})$ be the image of s. The basic theory of CM, [28, Theorem 3.13], asserts that, for a principally polarized CM abelian surface $\mathbf{A} = (A, \kappa_A, \lambda_A)$ over $\mathbb{C}$ of type Σ, the action of η coincides with the action of $\mathbf{t} = (tO_F, \xi)$ up to isomorphism, i.e.,

$$\eta(\mathbf{A}) \simeq \mathbf{t} \bullet \mathbf{A}.$$

Here note that $t \cdot \bar{t} = \xi \cdot \chi_{\mathrm{cyc}}(\eta) \in \widehat{\mathbb{Q}}^\times = \mathbb{Q}_+^\times \times \widehat{\mathbb{Z}}^\times$, where ξ is as above and $\chi_{\mathrm{cyc}} : \mathrm{Aut}(\mathbb{C}) \to \widehat{\mathbb{Z}}^\times$ is as in [28]. Thus the $\mathrm{Aut}(\mathbb{C}/E_\Sigma)$ orbits coincide with the orbits of the subgroup $N_\Sigma(\widehat{E}_\Sigma^\times)$ of $T_E(\widehat{\mathbb{Q}})$.

Next suppose that $\eta \in \mathrm{Aut}(\mathbb{C})$ does not restrict to the identity on E_Σ. Then there is an element $f_\Sigma(\eta) \in \widehat{E}^\times/E^\times$, the Tate cocycle, [28, section 4], such that $\eta(\mathbf{A}) \in \mathcal{CM}_E^{\eta\Sigma}(\mathbb{C})$ has the following description. Let $t \in \widehat{E}^\times$ be an element of the coset $f_\Sigma(\eta)$, and note that $t \cdot \bar{t} = \chi_{\mathrm{cyc}}(\eta) \cdot \xi$, where $\xi \in F^\times$, [28, Proposition 4.6 (b)]. Here χ_{cyc} is the cyclotomic character

$$\chi_{\mathrm{cyc}} : \mathrm{Aut}(\mathbb{C}) \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \hat{\mathbb{Z}}^\times.$$

Let $\mathfrak{a} = t\,O_E$ be the fractional ideal generated by t and note that $N_{E/F}\mathfrak{a} = \xi O_F$, so that $\mathbf{c} = (\mathfrak{a}, \xi)$ is a pair defining a class in $C(E)$. Note that $\widetilde{\mathrm{gen}}_f(\mathbf{c}) = \chi_{\mathrm{cyc}}(\eta)$. Then, by the extension of CM theory due to Tate, Langlands and Deligne, cf. [28, Theorem 4.1],

$$M(\eta(\mathbf{A})) \simeq \mathbf{c} \bullet M(\mathbf{A}), \tag{6.7}$$

where $M(\eta(\mathbf{A}))$ is the principally polarized CM module of $\eta(\mathbf{A})$. In particular, this module, which determines the isomorphism class of $\eta(\mathbf{A})$, has type $\eta\Sigma$. For example, let $\iota \in \mathrm{Aut}(\mathbb{C})$ be complex conjugation. Then $f_\Sigma(\iota) = 1$, [28, Proposition 4.8 (c)], and $\chi_{\mathrm{cyc}}(\iota) = -1$. Thus $M(\iota(\mathbf{A})) \simeq (M(A), \kappa(A), -\lambda_A)$; this module has type $\bar{\Sigma}$, and the corresponding complex structure on $M(A)_\mathbb{R}$ is $-\mathtt{h}_A$. Moreover, note that, by (ii) of Proposition 6.2, there is an isomorphism

$$(\widehat{W}(M(\eta(\mathbf{A}))), Q^\sharp_{M(\eta(\mathbf{A}))}) \xrightarrow{\sim} (\widehat{W}(M(\mathbf{A})), Q^\sharp_{M(\mathbf{A})}), \tag{6.8}$$

carrying $\widehat{L}(\eta(\mathbf{A}))$ to $\widehat{L}(\mathbf{A})$, since $\widetilde{\mathrm{gen}}_f(\mathbf{c}) = \chi_{\mathrm{cyc}}(\eta)$ implies that $N_{F/\mathbb{Q}}(\widetilde{\mathrm{gen}}_f(\mathbf{c})) \in (\widehat{\mathbb{Z}}^\times)^2$ is the square of a unit.

6.3 The forgetful morphism

There is a forgetful morphism

$$j_E^\Sigma : \mathcal{CM}_E^\Sigma \to \mathcal{M}/O_\Sigma \tag{6.9}$$

given by restricting the O_E action to an action of O_F, i.e., sending (A, κ_A, λ_A) to $(A, \kappa_A\,|_{O_F}, \lambda_A)$. Here $\mathcal{M}/O_\Sigma$ denotes the base change of $\mathcal{M}$ to $\mathrm{Spec}(O_\Sigma)$. We now describe the induced map of orbifolds $\mathcal{CM}_E^\Sigma(\mathbb{C}) \to \mathcal{M}(\mathbb{C})$ and its relation to the special 0-cycles of section 2. Fix $\mathbf{A} = (A, \kappa_A, \lambda_A)$, as before, and choose an isomorphism

$$(*): \qquad (M(A), \kappa_A\,|_{O_F}, \lambda_A) \xrightarrow{\sim} (\overset{o}{M}, \overset{o}{\kappa}, \overset{o}{\lambda}).$$

of the associated RM module with the standard one. This induces an isomorphism of $V(M(A))$ with V and of $\mathrm{GSpin}(V(M(A)))$ with $\mathrm{GSpin}(V)$. Moreover, for the $F^\sharp$-quadratic space $(W(M(A)), Q^\sharp_{M(A)})$, we have

$$\mathrm{Res}_{F^\sharp/\mathbb{Q}}(W(M(A))) \simeq V, \qquad \mathrm{tr}_{F^\sharp/\mathbb{Q}}(Q^\sharp_{M(A)}) = Q,$$

and the signature is given by Proposition 6.1. The torus T_E in $\mathrm{GSpin}(V(M(A)))$ and the extension (6.5) is identified with the torus T in $\mathrm{GSpin}(V)$ of Lemma 2.1 and the extension (2.4) associated to $W = W(M(A))$. Having fixed the isomorphism (*), let $z_\mathbf{A} = z(\mathtt{h}_\mathbf{A}) \in \mathbb{D}^+$ be the oriented negative 2-plane associated to $\mathtt{h}_\mathbf{A} \in \mathcal{D}$. Let $\underline{\mathtt{h}}_\mathbf{A}$ be the associated homomorphism of $\mathbb{S}$ into $G_\mathbb{R}$.

Proposition 6.4. (i) Under the forgetful morphism,

$$\begin{array}{ccccc} [\mu(E)\backslash C(T_E)] & \xrightarrow{\sim} & Z(C(T_E), \mathbf{A}) & \longrightarrow & \mathcal{M}(\mathbb{C}) \\ \| & & \downarrow\wr & & \downarrow\wr\ (**) \\ [T(\mathbb{Q})\backslash(\{\underline{\mathtt{h}}_\mathbf{A}\} \times T(\widehat{\mathbb{Q}})/K_T)] & = & Z(T, \underline{\mathtt{h}}_\mathbf{A}) & \longrightarrow & [G(\mathbb{Q})\backslash\mathbb{D} \times G(\widehat{\mathbb{Q}})/K] \end{array}$$

where $Z(T, \underline{\mathtt{h}}_{\mathbf{A}})$ is the cycle defined in section 2.

(ii) Let $\iota(\mathbf{A})$ be the transform of $\mathbf{A}$ under complex conjugation $\iota \in \mathrm{Aut}(\mathbb{C})$. Let $\underline{\mathtt{h}}_{\mathbf{A}}^{-}$ be the complex structure corresponding to $\bar{z}_{\mathbf{A}} \in \mathbb{D}^{-}$, the 2-plane $z_{\mathbf{A}}$ but with the opposite orientation. Then

$$\begin{array}{ccccc} [\mu(E)\backslash C(T_E)] & \overset{\sim}{\longrightarrow} & Z(C(T_E), \iota(\mathbf{A})) & \longrightarrow & \mathcal{M}(\mathbb{C}) \\ \| & & \downarrow \wr & & \downarrow \wr \; (**) \\ [T(\mathbb{Q})\backslash(\{\underline{\mathtt{h}}_{\mathbf{A}}^{-}\} \times T(\widehat{\mathbb{Q}})/K_T)] & == & Z(T, \underline{\mathtt{h}}_{\mathbf{A}}^{-}) & \longrightarrow & [G(\mathbb{Q})\backslash \mathbb{D} \times G(\widehat{\mathbb{Q}})/K] \end{array}$$

$\square$

Finally, let Σ' be a CM type for E distinct from Σ and $\bar{\Sigma}$, and take $\eta \in \mathrm{Aut}(\mathbb{C})$ such that $\Sigma' = \eta\Sigma$. Let $\mathbf{B} = \eta(\mathbf{A})$. Fixing an isomorphism (***) of $(M(B), \kappa_B\,|_{O_F}, \lambda_B)$ with $(\overset{o}{M}, \overset{o}{\kappa}, \overset{o}{\lambda})$, we obtain a point $\mathtt{h}_{\mathbf{B}} \in \mathcal{D}$, an isomorphism $V(B) \overset{\sim}{\longrightarrow} V = V(\overset{o}{M})$ of rational quadratic spaces, and an embedding $i_B : T_E \to G$. Let $\mathbf{c} = (tO_E, \xi)$ be the pair determined by η, as described above, and fix the isomorphism (6.7) of $M(\mathbf{B})$ and $\mathbf{c} \bullet M(\mathbf{A})$. By (6.8), we then obtain an isometry $\varphi : \widehat{W}(M(\mathbf{B})) \overset{\sim}{\longrightarrow} \widehat{W}(M(\mathbf{A}))$ of $\widehat{F}^{\sharp}$-quadratic spaces carrying $\widehat{L}(M(\mathbf{B}))$ to $\widehat{L}(M(\mathbf{A}))$. Combining these, we determine a unique element $g_0 \in \mathrm{SO}(V)(\widehat{\mathbb{Q}})$ such that the diagram

$$\begin{array}{ccc} V(\widehat{\mathbb{Q}}) & \overset{g_0^{-1}}{\longrightarrow} & V(\widehat{\mathbb{Q}}) \\ \| & & \| \\ \widehat{W}(M(\mathbf{B})) & \underset{\varphi}{\overset{\sim}{\longrightarrow}} & \widehat{W}(M(\mathbf{A})) \end{array}$$

commutes. Let T' be the torus $i_{\mathbf{B}}(T_E)$ in G. Note that, for $s \in T_E(\widehat{\mathbb{Q}})$, $i_{\mathbf{B}}(s) = g_0\, i_{\mathbf{A}}(s)\, g_0^{-1}$, and that $\widehat{L}(\mathbf{B}) = g_0^{-1}\widehat{L}(\mathbf{A})$.

Proposition 6.5. With the notation just introduced, for any element $g \in G(\widehat{\mathbb{Q}})$ with image g_0 in $\mathrm{SO}(V)(\widehat{\mathbb{Q}})$,

$$\begin{array}{ccccc} [\mu(E)\backslash C(T_E)] & \overset{\sim}{\longrightarrow} & Z(C(T_E), \mathbf{B}) & \longrightarrow & \mathcal{M}(\mathbb{C}) \\ \| & & \downarrow \wr & & \downarrow \wr \; (**) \\ [T'(\mathbb{Q})\backslash(\{\underline{\mathtt{h}}_{\mathbf{B}}\} \times T'(\widehat{\mathbb{Q}})/K_{T'})] & == & Z(T', \underline{\mathtt{h}}_{\mathbf{B}}, g) & \longrightarrow & [G(\mathbb{Q})\backslash \mathbb{D} \times G(\widehat{\mathbb{Q}})/K]. \end{array}$$

Here note that the uniformization (**) depends on the choice of base point $\mathbf{A}$. $\square$

Corollary 6.6. The morphism (6.9) induces an isomorphism of orbifolds

$$Z(C(T_E), \mathbf{A}) \,\cup\, Z(C(T_E), \iota(\mathbf{A})) \,\cup\, Z(C(T_E), \eta(\mathbf{A})) \,\cup\, Z(C(T_E), \iota\eta(\mathbf{A})) \overset{\sim}{\longrightarrow} Z(T, \mathtt{h}_{\mathbf{A}}^{\pm}) \,\cup\, Z(T', \mathtt{h}_{\eta(\mathbf{A})}^{\pm}, g).$$

$\square$

The 0-cycle on the right here was denoted by $Z(W)$ in section 2, where $W = W(M(\mathbf{A}))$. To lighten notation, we will now write $Z(\mathbf{A})$ for this cycle. Similarly, we will write $L(\mathbf{A})$ for $L(M(\mathbf{A}))$ for the lattice of special endomorphisms of $M(\mathbf{A})$.

Remark 6.7. A key point here is that the 0-cycle $j_E : \mathcal{CM}_E(\mathbb{C}) \to \mathcal{M}(\mathbb{C})$ associated to the non-biquadratic CM field E/F via moduli coincides with a union of 0-cycles associated to the binary quadratic spaces for the non-biquadratic CM fields $E^\sharp/F^\sharp$ via the Shimura variety construction of Section 2. □

Remark 6.8. Let $\bar{\mathbf{A}} = (A, \bar\kappa_A, \lambda_A) \in \mathcal{CM}_E^{\bar\Sigma}(\mathbb{C})$ where $\bar\kappa_A(a) = \kappa_A(\bar a)$. Note that the underlying RM modules $(M(A), \kappa_A\, |_{O_F}, \lambda_A)$ and $(M(A), \bar\kappa_A\, |_{O_F}, \lambda_A)$ coincide‡ and that, by (6.1), the complex structures $\mathtt{h}_{\mathbf{A}}$ and $\mathtt{h}_{\bar{\mathbf{A}}}$ also coincide. Thus, the images of $Z(C(T_E), \mathbf{A})$ and $Z(C(T_E), \bar{\mathbf{A}})$ in $\mathcal{M}(\mathbb{C})$ coincide, and both are identified with $Z(T, \underline{\mathtt{h}}_{\mathbf{A}})$ under the uniformization isomorphism (**). □

Combining Theorem 5.2 with Corollary 6.6, we obtain the following result.

Corollary 6.9. Let $f \in H_{0,\bar\rho_L}$, and let E be a non-biquadratic CM quartic field with real subfield F and a CM type Σ. Let

$$c(E) = \frac{4}{w_E} \frac{|C(T)|}{\Lambda(0, \chi)}.$$

(i) For $\mathbf{A} \in \mathcal{CM}_E^{\Sigma}(\mathbb{C})$,

$$\Phi(Z(\mathbf{A}), f) = c(E) \left(\mathrm{CT}[\langle f^+, \mathcal{E}(\tau, L(\mathbf{A}))\rangle] - \mathcal{L}'(0, \xi(f), L(\mathbf{A})) \right).$$

(ii) Thus

$$j^*\Phi(f)(\mathcal{CM}_E(\mathbb{C})) = c(E) \sum_{\mathbf{A} \in C(T_E)\backslash \mathrm{CM}^\Sigma(E)(\mathbb{C})} \left(\mathrm{CT}[\langle f^+, \mathcal{E}(\tau, L(\mathbf{A}))\rangle] - \mathcal{L}'(0, \xi(f), L(\mathbf{A})) \right).$$

where $\mathrm{CM}^\Sigma(E)(\mathbb{C})$ is the set of isomorphism classes of objects in $\mathcal{CM}_E^\Sigma(\mathbb{C})$. □

Note that Theorem 5.2 gives (i) here but with the constant

$$c_1(E) = \frac{\deg Z(W(\mathbf{A}), z_0^{\pm})}{\Lambda(0, \chi_\Sigma)}.$$

in place of $c(E)$, where χ_Σ is the quadratic Hecke character of F_Σ associated to E_Σ/F_Σ. By the proof of [42, Proposition 3.3], one has $\Lambda(s, \chi) = \Lambda(s, \chi_\Sigma)$, so $c_1(E) = c(E)$.

6.4 Integral Structure

Let $E = F(\sqrt{\Delta})$ be again a quartic CM number field with real quadratic subfield F. Let Σ be a CM type of E and let E_Σ be its reflex field.

‡And we use the same isomorphism (*) for both.

Hypothesis 6.10. From now on, we assume that $D \equiv 1 \mod 4$ is prime and $d_E = D^2\tilde{D}$ with $\tilde{D} \equiv 1 \mod 4$ squarefree. In this case, we can find Δ with $E = F(\sqrt{\Delta})$ such that

$$\Delta = \frac{a + b\sqrt{D}}{2}$$

is primitive in the sense that a and b are relatively prime rational integers (see [9, (4.20)] and [9, Lemma 4.4]). $\square$

For convenience, we will identify $E_\Sigma = \phi_\Sigma(E^\sharp)$ with $E^\sharp$ and its real quadratic subfield $F_\Sigma = \phi_\Sigma(F^\sharp) = \mathbb{Q}(\sqrt{\tilde{D}})$ with $F^\sharp$. We use the abbreviations $W(\mathbf{A}) = W(M(\mathbf{A}))$, $L(\mathbf{A}) = L(M(\mathbf{A}))$, and so on. Finally, we write $\mathrm{CM}^\Sigma(E)$ for $\mathcal{CM}_E^\Sigma(\mathbb{C})$.

We consider the $F^\sharp$-quadratic space

$$W^\sharp = E^\sharp, \quad Q^\sharp(z) = -\frac{z\bar{z}}{\sqrt{\tilde{D}}} \tag{6.10}$$

with the even integral lattice $L^\sharp = O_{E^\sharp}$. The main purpose of this subsection is to prove

Proposition 6.11. Let the notation and assumption be as above. Then for any $\mathbf{A} \in \mathrm{CM}^\Sigma(E)$, there is an $F^\sharp$-quadratic isomorphism

$$\phi_{\mathbf{A}} : (W(\mathbf{A}), Q^\sharp_{\mathbf{A}}) \cong (W^\sharp, Q^\sharp)$$

such that $\phi_{\mathbf{A}}(L(\mathbf{A}))$ is in the same genus as $L^\sharp$. In particular $L(\mathbf{A})$ is an $O_{E^\sharp}$-module and all $(L(\mathbf{A}), Q^\sharp_{\mathbf{A}})$ are in the same genus. $\square$

Proof. We divide the proof in to several steps.

Step 1: Preparation. To prove Proposition 6.11, we need a concrete model of $\mathbf{A} \in \mathrm{CM}^\Sigma(E)$. Let $\mathfrak{A}$ be a fractional ideal of E and $\xi \in E$ such that

$$\bar{\xi} = -\xi, \quad \Sigma(\xi) = (\sigma_1(\xi), \sigma_2(\xi)) \in \mathbb{H}^2, \quad \left(\xi \partial_{E/F} \mathrm{N}_{E/F} \mathfrak{A} \right) \cap F = \partial_F^{-1}. \tag{6.11}$$

Here $\partial_{E/F}$ is the relative different of E/F. Then there is an CM abelian surface $\mathbf{A} = \mathbf{A}(\mathfrak{A}, \xi) \in \mathrm{CM}^\Sigma(E)$ associated to it, given as follows. First, $A = \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{R}/\mathfrak{A}$ with O_E-multiplication on the left as the O_E-action κ, second the following symplectic form

$$\lambda = \lambda_\xi : \mathfrak{A} \times \mathfrak{A} \to \mathbb{Z}, \quad (x, y) \mapsto \mathrm{tr}_{E/\mathbb{Q}}(\xi \bar{x} y)$$

gives rise to a Riemmann form on A and thus a polarization λ, which is principal. Conversely, every principally polarized CM abelian surface $\mathbf{A} \in \mathrm{CM}^\Sigma(E)$ can be constructed this way (see for example [9, Section 3]). Let

$\mathbf{A} = \mathbf{A}(\mathfrak{A}, \xi) \in \mathrm{CM}^\Sigma(E)$. By [9, Lemma 3.2] or direct checking, one can choose α and β such that

$$\mathfrak{A} = O_F\alpha + \partial^{-1}\beta, \quad \xi\,(\bar\alpha\beta - \alpha\bar\beta) = 1. \tag{6.12}$$

Then we put

$$f := f_{\alpha,\beta} : \mathfrak{A} \to {}^t(O_F \oplus \partial^{-1}), \quad x\alpha + y\beta \mapsto \begin{pmatrix} x \\ y \end{pmatrix},$$

and define

$$\kappa := \kappa_{\alpha,\beta} : O_E \to \mathrm{End}_{O_F}\, {}^t(O_F \oplus \partial^{-1}) \subset M_2(F),$$

by

$$(r\alpha, r\beta) \begin{pmatrix} x \\ y \end{pmatrix} = (\alpha, \beta)\,\kappa(r) \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then $f_{\alpha,\beta}$ gives an isomorphism between the principally polarized RM modules $M(\mathbf{A})$ and $(\overset{o}{M}, \overset{o}{\kappa}, \overset{o}{\lambda})$. Recall that we can and will identity $L(\overset{o}{M})$, and thus $L(\mathbf{A})$, with

$$L = \left\{ \begin{pmatrix} a & b\sqrt{D} \\ c/\sqrt{D} & \sigma(a) \end{pmatrix} :\, a \in O_F, b, c \in \mathbb{Z} \right\},$$

and $V(\overset{o}{M})$, and thus $V(\mathbf{A})$, with $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. Notice that $W(\mathbf{A}) = V(\mathbf{A})$ as vector spaces over $\mathbb{Q}$.

Step 2: Define an isomorphism

$$\phi : W(\mathbf{A}) = V \xrightarrow{\sim} E^\sharp, \quad \phi(A) = \frac{1}{\sqrt{D}}(\sigma_1(\alpha), \sigma_1(\beta))Aw(\sigma_2(\alpha), \sigma_2(\beta))^t, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We prove that ϕ is an isomorphism of quadratic spaces over $F^\sharp$ between $(W(\mathbf{A}), Q^\sharp_{\mathbf{A}}))$ and $(E^\sharp, -\frac{z\bar z}{\sqrt{\tilde D}\,\mathrm{N}(\mathfrak{A})})$. To verify the claim, let

$$V_1 = \left\{A \in M_2(F) :\, \sigma(A) = A^t\right\} = \left\{ \left(\begin{smallmatrix} b & u \\ \sigma(u) & a \end{smallmatrix}\right) :\, a, b \in \mathbb{Q},\, u \in F \right\} \tag{6.13}$$

with the quadratic form $Q_1(A) = D \det A$, and let

$$\phi_1 : V \to V_1, \quad A \mapsto \frac{1}{\sqrt{D}} Aw, \tag{6.14}$$

$$\phi_2 : V_1 \to E^\sharp, \quad A \mapsto (\sigma_1(\alpha), \sigma_1(\beta))A(\sigma_2(\alpha), \sigma_2(\beta))^t. \tag{6.15}$$

Then $\phi = \phi_2 \circ \phi_1$. It is easy to check that ϕ_1 is a $\mathbb{Q}$-isomorphism between $(V, \det)$ and $(V_1, D\det)$. On the other hand, ϕ_2 is basically the map in [9, (4.11)], and is a $\mathbb{Q}$-isomorphism between $(V_1, D\det)$ and $(E^\sharp, -\operatorname{tr}_{F^\sharp/\mathbb{Q}} \frac{z\bar{z}}{\sqrt{\tilde{D}}\,\mathrm{N}(\mathfrak{A})})$. So ϕ is a $\mathbb{Q}$-quadratic space isomorphism. Next, For $r^\sharp = \mathrm{N}_\Sigma(r) \in E^\sharp$ with $r \in E^\times$, one has

$$\begin{aligned}
\phi(r^\sharp \bullet A) &= \frac{1}{\sqrt{D}}(\sigma_1(\alpha), \sigma_1(\beta))\kappa(r)A\sigma(\kappa(r)^\iota)w(\sigma_2(\alpha), \sigma_2(\beta))^t \\
&= \frac{1}{\sqrt{D}}(\sigma_1(r\alpha), \sigma_1(r\beta))Aw\sigma(\kappa(r)^\iota)^t(\sigma_2(\alpha), \sigma_2(\beta))^t \\
&= \frac{1}{\sqrt{D}}(\sigma_1(r\alpha), \sigma_1(r\beta))Aw(\sigma_2(r\alpha), \sigma_2(r\beta))^t \\
&= \sigma_1(r)\sigma_2(r)\phi(A) = r^\sharp\phi(A).
\end{aligned}$$

So ϕ is $E^\sharp$-linear. So ϕ is an $F^\sharp$-quadratic space isomorphism between $(W(\mathbf{A}), Q^\sharp_{\mathbf{A}})$ and $(E^\sharp, -\frac{z\bar{z}}{\sqrt{\tilde{D}}\,\mathrm{N}(\mathfrak{A})})$, as claimed.

Step 3: Let

$$L^0(\partial_F^{-1}) = \left\{ \begin{pmatrix} b & \lambda \\ \sigma(\lambda) & a \end{pmatrix} \in V_1 : a \in \frac{1}{D}\mathbb{Z},\, b \in \mathbb{Z}, \lambda \in \partial^{-1} \right\}$$

be a lattice in $(V_1, D\det)$. Then, by [9, Proposition 4.7], one has

$$\begin{aligned}
\phi(L(\mathbf{A})) &= \phi_2\phi_1(L) \\
&= \phi_2(L^0(\partial_F^{-1})) \\
&= \mathrm{N}_\Sigma(\mathfrak{A}).
\end{aligned}$$

Here $\mathrm{N}_\Sigma(\mathfrak{A})$ is the type norm of $\mathfrak{A}$ defined as

$$\mathrm{N}_\Sigma(\mathfrak{A}) = \sigma_1(\mathfrak{A})\sigma_2(\mathfrak{A})O_M \cap E^\sharp$$

for any Galois extension M of $\mathbb{Q}$ containing both E and $E^\sharp$. Thus, $\phi(L(\mathbf{A}))$ is actually a fractional ideal in $E^\sharp$, and in particular an $O_{F^\sharp}$-lattice, and we have

$$\phi : (L(\mathbf{A}), Q^\sharp_{\mathbf{A}}) \cong (\mathrm{N}_\Sigma(\mathfrak{A}), -\frac{z\bar{z}}{\sqrt{\tilde{D}}\,\mathrm{N}(\mathfrak{A})}). \tag{6.16}$$

Step 4: We prove that for every $\mathbf{A} = \mathbf{A}(\mathfrak{A}, \xi) \in \mathrm{CM}^\Sigma(E)$, one has for every finite prime $\mathfrak{p}$ of $F^\sharp$

$$(\mathrm{N}_\Sigma(\mathfrak{A})_\mathfrak{p}, -\frac{1}{\sqrt{\tilde{D}}}\frac{z\bar{z}}{\mathrm{N}(\mathfrak{A})}) \cong (L^\sharp_\mathfrak{p}, Q^\sharp). \tag{6.17}$$

Notice that ([9, Corollary 4.5])

$$\mathrm{N}_{E^\sharp/\mathbb{Q}}\,\partial_{E^\sharp/F^\sharp} = D, \quad \mathrm{N}_{E^\sharp/F^\sharp}(\mathrm{N}_\Sigma(\mathfrak{A})) = \mathrm{N}(\mathfrak{A})O_{F^\sharp}. \tag{6.18}$$

In particular, $E^\sharp/F^\sharp$ is ramified at exactly one prime $\mathfrak{D}$ of $F^\sharp$ and this prime $\mathfrak{D}$ is above D. For each prime ideal $\mathfrak{p} \neq \mathfrak{D}$ of $F^\sharp$, there is a generator α of $(\mathrm{N}_\Sigma(\mathfrak{A}))_\mathfrak{p}^\times$ such that $\alpha\bar{\alpha} = \mathrm{N}(\mathfrak{A})$. So $r \mapsto r/\alpha$ gives

$$((\mathrm{N}_\Sigma \mathfrak{A})_\mathfrak{p}, -\frac{1}{\sqrt{\tilde{D}}}\frac{z\bar{z}}{\mathrm{N}(\mathfrak{A})}) \cong (O_{E^\sharp,\mathfrak{p}}, -\frac{1}{\sqrt{\tilde{D}}}z\bar{z}) = (L_\mathfrak{p}^\sharp, Q^\sharp).$$

For $\mathfrak{p} = \mathfrak{D}$, one has similarly,

$$(\mathrm{N}_\Sigma(\mathfrak{A}), -\frac{z\bar{z}}{\sqrt{\tilde{D}}\,\mathrm{N}(\mathfrak{A})}) \cong (O_{E^\sharp,\mathfrak{D}}, -c\frac{1}{\sqrt{\tilde{D}}}z\bar{z})$$

for some some $c \in O_{F^\sharp,\mathfrak{D}}^\times$ with

$$\mathrm{N}(\mathfrak{A}) = c\alpha\bar{\alpha}, \quad \alpha \in \mathrm{N}_\Sigma(\mathfrak{A})_\mathfrak{D}.$$

Notice that $(W(\mathbf{A}), Q_\mathbf{A}^\sharp) \cong (E^\sharp, -\frac{1}{\sqrt{\tilde{D}}}\frac{z\bar{z}}{\mathrm{N}(\mathfrak{A})})$ and $(W^\sharp, Q^\sharp)$ are global $F^\sharp$-vector spaces, which are isomorphic at every prime $\mathfrak{p} \neq \mathfrak{D}$ by the above argument. So they have to be isomorphic at $\mathfrak{D}$ too, which implies that $c \in \mathrm{N}_{E_\mathfrak{D}^\sharp/F_\mathfrak{D}^\sharp}(O_{E^\sharp,\mathfrak{D}}^\times)$, and one has again (6.17) for the prime $\mathfrak{p} = \mathfrak{D}$. Finally, $W(\mathbf{A})$ and $W^\sharp$ are clearly isomorphic at all infinite places of $F^\sharp$ by (6.16). So there is an $\mathbb{A}_{F^\sharp}$ isomorphism of $\mathbb{A}_{F^\sharp}$-quadratic spaces

$$\phi'_\mathbf{A}: \quad (W(\mathbf{A})_\mathbb{A}, Q_\mathbf{A}^\sharp) \cong (W_\mathbb{A}^\sharp, Q^\sharp)$$

such that $\phi'_\mathbf{A}(\hat{L}(\mathbf{A})) = \hat{L}^\sharp$. By the Hasse principle, one proves the proposition. ■

Remark 6.12. Once one knows, say by the explicit calculations above, that $L(\mathbf{A})$ is actually an $O_{E^\sharp}$-module, one can prove without explicit computation that the genus of $(L(\mathbf{A}), Q_\mathbf{A}^\sharp)$ does not depend on the choice of $\mathbf{A} \in \mathrm{CM}^\Sigma(E)$. The basic idea is the following fact ([17, Lemma 2.4.4]). When $\mathbf{B} = \mathbf{c} \bullet \mathbf{A}$ with $\mathbf{c} = (\mathfrak{a}, \xi) \in C^+(E)$, there is a $\hat{F}^\sharp$-isometry

$$\phi : (\hat{W}(\mathbf{B}), Q_B^\sharp) \cong (\hat{W}(\mathbf{A}), \mathrm{N}_{F/\mathbb{Q}}(z_f)Q_\mathbf{A}^\sharp),$$

which sends $\hat{L}(\mathbf{B})$ onto $\hat{L}(\mathbf{A})$. Here $z_f = \xi^{-1}t\bar{t} \in \hat{O}_F^\times$ corresponds to the finite genus $\mathrm{gen}_f(\mathbf{c})$ of $\mathbf{c}$ and $t \in \hat{E}^\times$ with $tO_E = \mathfrak{a}$. One uses again the fact that $E^\sharp/F^\sharp$ ramifies at only one prime $\mathfrak{D}$ (see last step of the proof of Proposition 6.11) to show $\mathrm{N}_{F/\mathbb{Q}}(z_f) \in \mathrm{N}_{E^\sharp/\mathbb{F}^\sharp}(\hat{O}_{E^\sharp}^\times)$. This proves the claim. Note, however, that the explicit calculation gives more information. Most importantly, it gives a concrete model for $L(\mathbf{A})$, with which one can explicitly compute the Eisenstein series and thus the right hand side of the main formula in Corollary 6.9. We will use the explicit model to compare our formula with the main result of [9] in the next subsection.

□

Let

$$E^*(\vec{\tau}, s, L^\sharp, \mathbf{1}) = \sum_{\mu \in L^{\sharp,\prime}/L^\sharp} E^*(\vec{\tau}, s, \varphi_\mu, \mathbf{1})\, \varphi_\mu \tag{6.19}$$

be the incoherent Eisenstein series associated to $L^\sharp$, and let $\mathcal{E}(\tau, L^\sharp)$ be the holomorphic part of $E^{*,\prime}(\tau^\Delta, 0, L^\sharp)$ with $\tau \in \mathbb{H}$. Note that

$$L^{\sharp,\prime}/L^\sharp \simeq \partial_{E^\sharp/F^\sharp}^{-1}/O_{E^\sharp} \simeq \mathbb{Z}/D\mathbb{Z}.$$

Proof of Theorem 1.2: Now Theorem 1.2 is a direct consequence of Corollary 6.9 and Proposition 6.11.

6.5 Scalar modular forms

In this subsection, we again assume that $d_E = D^2\tilde{D}$ with $D \equiv 1 \mod 4$ prime and $\tilde{D} \equiv 1 \mod 4$ square-free. We translate Theorem 1.2 into the usual language of scalar modular forms and finally compare it with [9, Theorem 1.4] in the special case considered there. Let $\mathbb{H}$ be the upper half plane. For $(z_1, z_2) \in \mathbb{H}^2$, let

$$w(z_1, z_2) = \begin{pmatrix} 1 & \\ & \sqrt{D} \end{pmatrix}^{-1} \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \sqrt{D} \end{pmatrix} \in V(\mathbb{C}),$$

and let $U(z_1, z_2)$ be the oriented negative 2-plane in $V(\mathbb{R})$ whose complexification $U(z_1, z_2)(\mathbb{C})$ is spanned by $w(z_1, z_2)$ and $\overline{w(z_1, z_2)}$ and whose orientation is give by $w(z_1, z_2) \wedge \overline{w(z_1, z_2)}$. This gives an identification of $\mathbb{H}^2$ with $\mathbb{D}^+$, equivariant for the action of $\mathrm{SL}_2(O_F) \simeq \Gamma$. Under the resulting identification

$$X = X_K(\mathbb{C}) \simeq \mathrm{SL}_2(O_F)\backslash\mathbb{H}^2,$$

the Hirzebruch-Zagier divisor T_n defined in [9] is related to the special divisor $Z(m, \mu)$ via

$$T_n = \frac{1}{2} \begin{cases} Z(\frac{n}{D}, 0) & \text{if } D|n, \\ Z(\frac{n}{D}, \mu) + Z(\frac{n}{D}, -\mu) & \text{if } D \nmid n. \end{cases} \tag{6.20}$$

Here, in the second case, $\mu \in L^{\sharp,\prime}/L^\sharp$ is determined by the condition that $Q(\mu) \equiv \frac{n}{D} \mod 1$. Let k be an even integer, and let $A_{k,\rho}$ be the space of real analytic modular forms of weight k with representation ρ, where $\rho = \rho_L$ or $\bar{\rho}_L$. Let $A_k^+(D, (\frac{D}{\cdot}))$ be the space of real analytic modular forms $f_{\mathrm{sc}}(\tau) = \sum_n a(n, v) q^n$ of weight k for the group $\Gamma_0(D)$ with character $(\frac{D}{\cdot})$ such that $a(n, v) = 0$ whenever $(\frac{D}{n}) = -1$. Here we use f_{sc} to denote a scalar valued modular form to distinguish it from vector valued modular forms in this paper. Then the following lemma is proved in [5].

Lemma 6.13. There is an isomorphism of vector spaces $A_{k,\rho} \to A_k^+(D, (\frac{D}{\cdot}))$,

$$f = \sum_{\mu \in L'/L} f_\mu \varphi_\mu \mapsto f_{\mathrm{sc}} = D^{\frac{1-k}{2}} f_0 | W_D.$$

The inverse map is given by

$$f_{\mathrm{sc}} \mapsto f = \frac{1}{2} D^{\frac{k-1}{2}} \sum_{\gamma \in \Gamma_0(D) \backslash \mathrm{SL}_2(\mathbb{Z})} (f_{\mathrm{sc}} | W_D | \gamma)\, \rho(\gamma)^{-1} \varphi_0,$$

where $W_D = \left(\begin{smallmatrix} 0 & -1 \\ D & 0 \end{smallmatrix}\right)$ denotes the Fricke involution. Moreover, if $f_{\mathrm{sc}}(\tau) = \sum_n a(n,v) q^n$, then f has the Fourier expansion

$$f = \frac{1}{2} \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Z} \\ n \equiv DQ(\mu)\ (D)}} \tilde{a}(n,v)\, q^{n/D}\, \phi_\mu,$$

where $\tilde{a}(n,v) = a(n,v)$ if $n \not\equiv 0 \mod D$, and $\tilde{a}(n,v) = 2\, a(n,v)$ if $n \equiv 0 \mod D$. □

In particular, the constant term of f_{sc} agrees with the constant term of f in the φ_0-component. The isomorphisms of Lemma 6.13 take harmonic weak Maass forms to harmonic weak Maass forms, (weakly) holomorphic modular forms to (weakly) holomorphic modular forms, and cusp forms to cusp forms.

Let

$$E^*_{\mathrm{sc}}(\tau, s) = \frac{1}{\sqrt{D}}\, E^*(\tau^\Delta, s, \phi_0, \mathbf{1}) | W_D \tag{6.21}$$

be the scalar image of $E^*(\tau^\Delta, s, L^\sharp, \mathbf{1})$, and let $\mathcal{E}_{\mathrm{sc}}(\tau)$ be the holomorphic part of

$$\tilde{f}(\tau) = \frac{d}{ds} E^*_{\mathrm{sc}}(\tau, s)|_{s=0}.$$

Then $\tilde{f}(\tau)$ is the function defined in [9, (7.2)]. By [9, Theorem 7.2], we have the following lemma, which reveals the nature of the coefficients in the q-expansion of $\mathcal{E}_{\mathrm{sc}}(\tau)$.

Lemma 6.14. Let the notation be as above. Then

$$\mathcal{E}_{\mathrm{sc}}(\tau) = -2\Lambda'(0, \chi) - 4 \sum_{m \in \mathbb{Z}_{>0}} b_m q^m$$

with

$$b_m = \sum_{\substack{t = \frac{n + m\sqrt{\tilde{D}}}{2D} \in d^{-1}_{E^\sharp/F^\sharp} \\ |n| < m\sqrt{\tilde{D}}}} B_t.$$

Here

$$B_t = (\mathrm{ord}_{\mathfrak{l}}(t) + 1) \cdot \rho(t d_{E^\sharp/F^\sharp} \mathfrak{l}^{-1}) \cdot \log \mathrm{N}(\mathfrak{l})$$

if there is a unique prime ideal $\mathfrak{l}$ of $F^\sharp$ with $\tilde{\chi}_\mathfrak{l}(t) = -1$, and otherwise $B_t = 0$. Here $\tilde{\chi}$ is the quadratic Hecke character of $F^\sharp$ associated to $E^\sharp/F^\sharp$. Finally

$$\rho(\mathfrak{a}) = |\{\mathfrak{A} \subset O_{E^\sharp} : \mathrm{N}_{E^\sharp/F^\sharp} \mathfrak{A} = \mathfrak{a}\}|.$$

□

Now let $f_{\mathrm{sc}} = f_{\mathrm{sc}}^+ + f_{\mathrm{sc}}^- \in H_0^+(D, (\frac{D}{\cdot}))$ be a harmonic weak Maass form with holomorphic part

$$f_{\mathrm{sc}}^+(\tau) = \sum_{n \gg -\infty} c^+(n) q^n,$$

and let

$$\tilde{c}^+(n) = \begin{cases} 2c^+(n) & \text{if } D|n, \\ c^+(n) & \text{if } D \nmid n. \end{cases}$$

Let $f \in H_{0,\bar{\rho}_{L^\sharp}}$ be the associated vector valued harmonic weak Maass form. We define a divisor on X and a Green function associated to f_{sc} by

$$T(f_{\mathrm{sc}}) = \sum_{n>0} \tilde{c}^+(-n) T_n, \tag{6.22}$$

$$\Phi(z, f_{\mathrm{sc}}) = \Phi(z, f). \tag{6.23}$$

Then one sees that $T(f_{\mathrm{sc}}) = Z(f)$ by (6.20). We also define the Rankin-Selberg L-series

$$\mathcal{L}_{\mathrm{sc}}(s, \xi(f_{\mathrm{sc}})) = \langle E_{\mathrm{sc}}^*(\tau, s), \xi(f_{\mathrm{sc}}) \rangle_{\mathrm{Pet}}. \tag{6.24}$$

A straightforward calculation gives

Lemma 6.15. (1)

$$\mathcal{L}(s, \xi(f), L^\sharp) = \frac{1}{2} D(D+1) \mathcal{L}_{\mathrm{sc}}(s, \xi(f_{\mathrm{sc}})).$$

(2)

$$\mathrm{CT}[\langle f^+, \mathcal{E}(\tau, L^\sharp) \rangle] = -2c^+(0)\Lambda'(0, \chi) - 2 \sum_{n>0} \tilde{c}^+(-n) b_n.$$

□

Combining this with Theorem 1.2, we obtain:

Corollary 6.16. Let $F = \mathbb{Q}(\sqrt{D})$ with $D \equiv 1 \mod 4$ prime, and let E be a CM non-biquadratic field with absolute discriminant $d_E = D^2\tilde{D}$ where $\tilde{D} \equiv 1 \mod 4$ is square free. If $f_{\rm sc} \in H_0^+(D, (\frac{D}{\cdot}))$, then

$$\Phi(\mathrm{CM}(E), f_{\rm sc}) = -2c'(E)\left[\sum_{n>0} \tilde{c}^+(-n)b_n + c^+(0)\Lambda'(0,\chi) + \frac{D(D+1)}{4}\mathcal{L}'_{\rm sc}(0, \xi(f_{\rm sc}))\right].$$

□

Now we assume that $f_{\rm sc} = \sum c^+(n)q^n \in H_0^+(D, (\frac{D}{\cdot}))$ is weakly holomorphic, i.e., $\xi(f_{\rm sc}) = 0$, and that $\tilde{c}^+(n) \in \mathbb{Z}$ for $n < 0$. Then there is a (up to a constant of modulus 1 unique) memomorphic Hilbert modular form $\Psi(z, f_{\rm sc})$ of weight $c^+(0)$ with a Borcherds product expansion whose divisor is given by

$$\mathrm{div}(\Psi) = T(f_{\rm sc}),$$

see [5, Theorem 9]. Morever, by construction it satisfies

$$-\log \|\Psi(z, f_{\rm sc})\|^2_{\rm Pet} = \Phi(z, f_{\rm sc}),$$

where

$$\|\Psi(z_1, z_2, f_{\rm sc})\|^2_{\rm Pet} = |\Psi(z_1, z_2, f_{\rm sc})|^2 (4\pi e^{-\gamma} y_1 y_2)^{c^+(0)}$$

is the Petersson metric (normalized in a way which is convenient for our purposes), and $\gamma = -\Gamma'(1)$ is Euler's constant.

Corollary 6.17. Let the notation be as in Corollary 6.16 and assume that $f_{\rm sc}$ is weakly homomorphic. Then

$$\log \|\Psi(\mathrm{CM}(E), f_{\rm sc})\|_{\rm Pet} = c'(E)\sum_{n>0} \tilde{c}^+(-n)b_n + c'(E)c^+(0)\Lambda'(0,\chi).$$

□

When $\tilde{D}$ is also prime, we have $c'(E) = 1$. Then this corollary coincides with [9, Theorem 1.4], since the CM points in this paper are counted with multiplicity $\frac{2}{w_E}$, and our CM cycle is twice the CM cycle there as a set with multiplicities.

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