Topological Dynamics

Probing dynamical systems using loops

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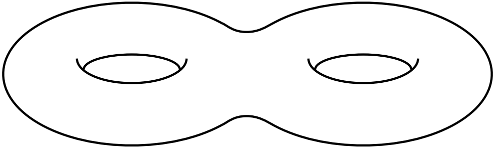
Surface dynamics

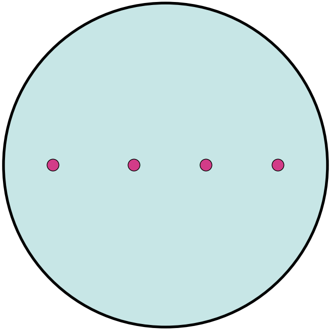
Low-dimensional topologists have long studied transformations of surfaces such as the double-torus:



The central object of study is the homeomorphism: a continuous, invertible transformation whose inverse is also continuous.

Punctured disks

A surface of more practical relevance is the punctured disk:



For instance, it is a model of a two-dimensional vat of viscous fluid with stirring rods.

Punctured disks in experiments

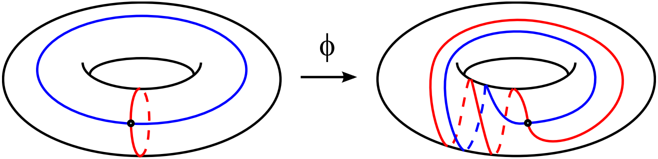
The transformation in this case is given by the solution of a fluid equation over one period of rod motion.

[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

[movie 1] [movie 2]

Action on curves

If we don't know anything about a transformation ϕ, we can learn a lot by looking at its action on some representative curves:



This is the action of the famous cat map of Arnold. In the language of topology we are looking at its action on the fundamental group.

Note that since the curves initially intersect only once, their image only intersects once as well.

Growth of curves for Cat Map

The Cat Map can be written

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pmod 1$$

where the torus is the bi-periodic domain $[0,1]^2$.

At each application of the map, curves grow asymptotically by a factor given by the largest eigenvalue of the matrix,

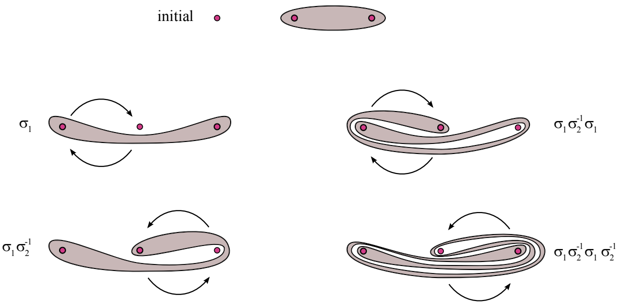
$$\lambda = \tfrac{1}{2}(3+\sqrt{5}) = \varphi^2, \qquad \varphi := \tfrac{1}{2}(1+\sqrt{5})$$

where φ is the Golden Ratio.

The rate of growth $h = \log \lambda$ is called the topological entropy.

Growth of curves on a disk

On a disk with 3 punctures (rods), we can also look at the growth of curves:



We use the braid generator notation: σ_i means the clockwise interchange of the ith and $(i+1)$th rod. (Inverses are counterclockwise.)

The motion above is denoted $\sigma_1\sigma_2^{-1}$.

Growth of curves on a disk (2)

But how do we find the rate of growth of curves for motions on the disk?

For 3 punctures it's easy: the entropy for $\sigma_1\sigma_2^{-1}$ is $h = \log\phi^2$, just like the Cat Map!

(This is not a coincidence: there is an intimate connection between the two, which we won't get into. But for the specialist: the key word is double cover.)

For more punctures, this is a hard problem.

Entropy calculation

The problem: given a periodic motion of n punctures on a disk, what is the entropy?

Many aproaches available:

- Interval exchange map (orientable foliations — not general enough);
- Train tracks and Bestvina–Handel algorithm (1995) (computationally very hard — overkill);
- Burau representation (Kolev, 1989): super-fast, but only a lower bound;
- Moussafir iterative technique (2006): fast and intuitive!

The Moussafir technique allows us to tackle large-scale problems.

Iterating a loop

It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

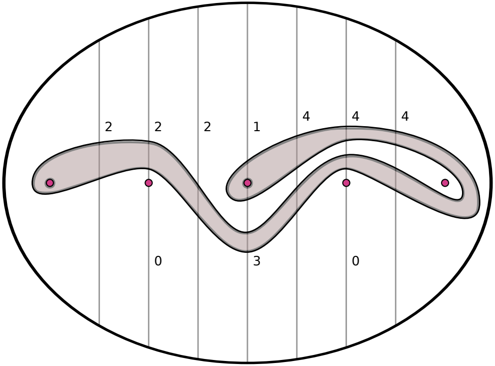
The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;
2. Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them topologically with very few numbers.

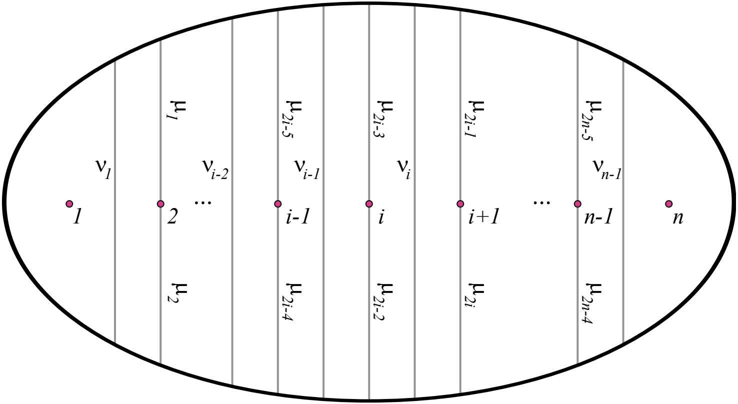
Solution to problem 1: Loop coordinates

What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the Dynnikov coordinates involve intersections with vertical lines:



Crossing numbers

Label the crossing numbers:



Dynnikov coordinates

Now take the difference of crossing numbers:

$$a_i = \frac{1}{2}\left(\mu_{2i} - \mu_{2i-1}\right),$$
$$b_i = \frac{1}{2}\left(\nu_i - \nu_{i+1}\right)$$

for $i = 1, \ldots, n-2$.

The vector of length $(2n-4)$,

$$\mathbf{u} = (a_1, \ldots, a_{n-2}, b_1, \ldots, b_{n-2})$$

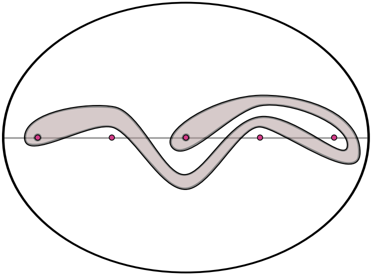
is called the Dynnikov coordinates of a loop.

There is a one-to-one correspondence between closed loops and these coordinates: you can't do it with fewer than $2n-4$ numbers.

Intersection number

A useful formula gives the minimum intersection number with the 'horizontal axis':

$$L(\mathbf{u}) = |a_1| + |a_{n-2}| + \sum_{i=1}^{n-3} |a_{i+1} - a_i| + \sum_{i=0}^{n-1} |b_i| \,,$$

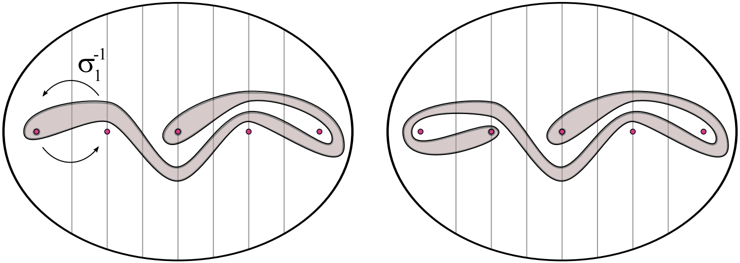


For example, the loop on the left has $L = 12$.

The crossing number grows proportionally to the the length.

Solution to problem 2: Action on coordinates

Moving the punctures according to a braid generator changes some crossing numbers:



There is an explicit formula for the change in the coordinates!

Action on loop coordinates

The update rules for σ_i acting on a loop with coordinates $(\mathbf{a}, \mathbf{b})$ can be written

$$\begin{aligned} a'_{i-1} &= a_{i-1} + b^+_{i-1} + \left(b^+_i - c_{i-1}\right)^+ , \\ b'_{i-1} &= b_i - c^+_{i-1} , \\ a'_i &= a_i + b^-_i + \left(b^-_{i-1} + c_{i-1}\right)^- , \\ b'_i &= b_{i-1} + c^+_{i-1} , \end{aligned}$$

where

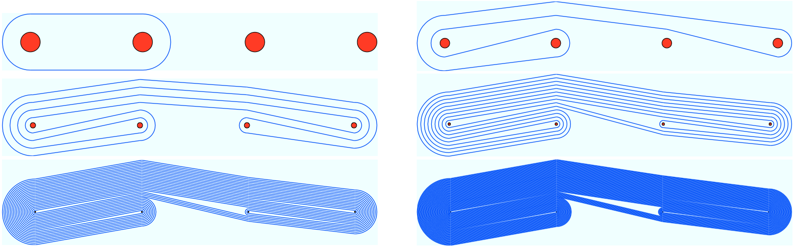
$$f^+ := \max(f, 0), \qquad f^- := \min(f, 0).$$

$$c_{i-1} := a_{i-1} - a_i + b^+_i - b^-_{i-1} ,$$

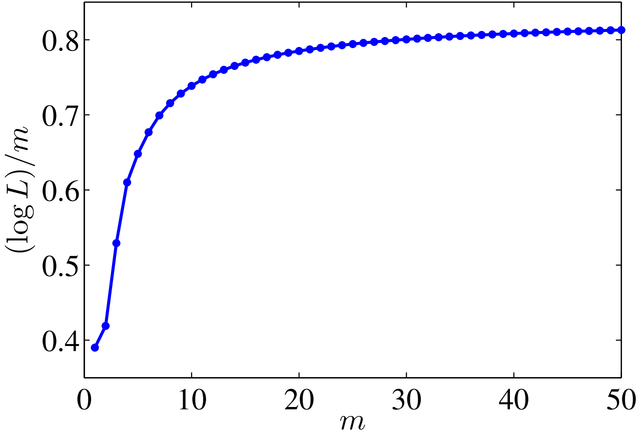
This is called a piecewise-linear action.
Easy to code up (see for example Thiffeault (2009)).

Growth of L

For a specific rod motion, say as given by the braid $\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1$, we can easily see the exponential growth of L and thus measure the entropy:



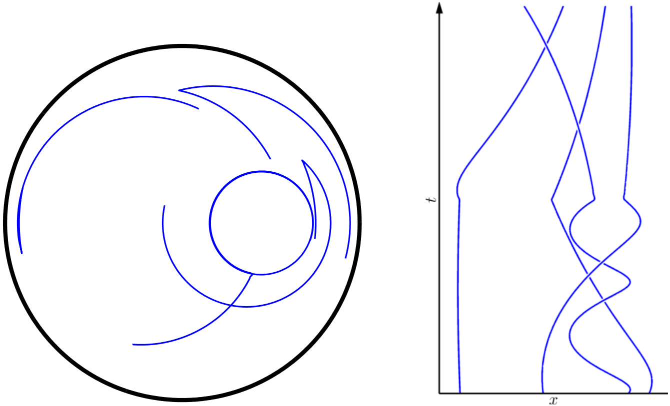
Growth of L (2)



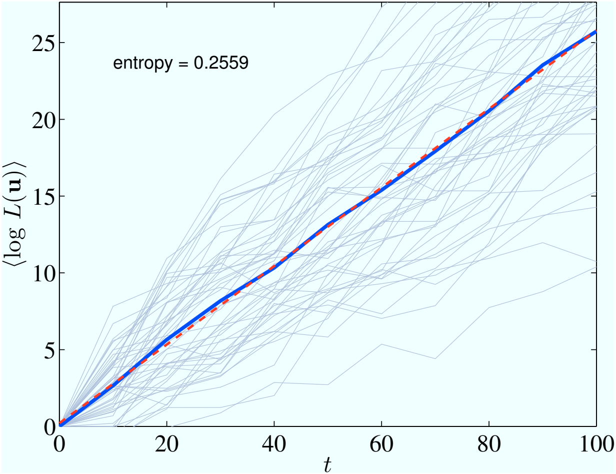
m is the number of times the braid acted on the initial loop.

Random particle trajectories

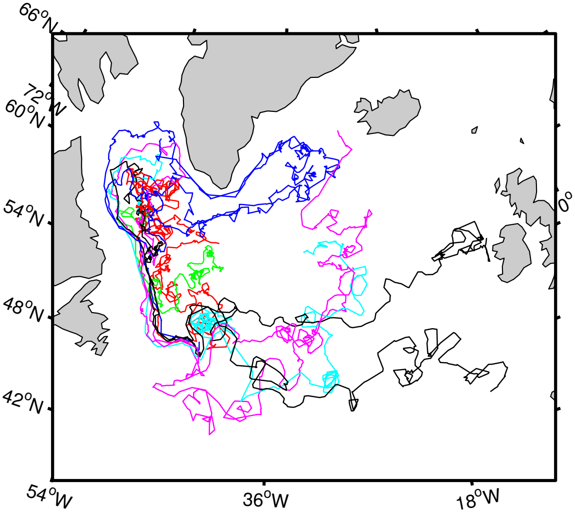
Now consider a set of n particles advected by some flow, such as the blinking vortex flow:



Entropy by averaging over trajectories



Oceanic float trajectories



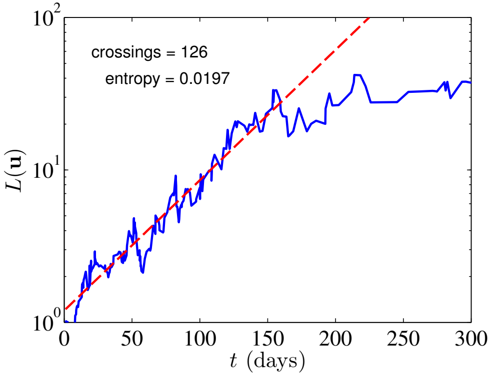
Oceanic floats: Data analysis

What can we measure?

- Single-particle dispersion (not a good use of all data)
- Correlation functions (what do they mean?)
- Lyapunov exponents (some luck needed!)

Oceanic floats: Entropy

10 floats from Davis' Labrador sea data:



Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

Source: WOCE subsurface float data assembly center (2004)

Conclusions

- Having rods undergo 'braiding' motion guarantees a minimal amount of entropy (stretching of material lines);
- This idea can also be used on fluid particles to estimate entropy;
- Need a way to compute entropy fast: loop coordinates;
- There is a lot more information in this braid: extract it! (keyword: Lagrangian coherent structures);
- Long-term goal: a toolbox of topological methods to analyze and make prediction about general flow properties;
- Holy grail: Three dimensions! (though current work applies to many 3D situations. . .)
- Preprint: http://arxiv.org/abs/0906.3647

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