A Topological Theory of Stirring

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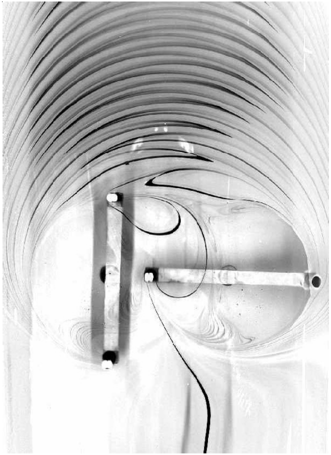
Figure-eight stirring protocol

- Classic stirring method!
- Viscous (Stokes) flow;
- Essentially two-dimensional;
- Two regular islands: there are effectively 3 rods!
- We call these Ghost Rods
- 'Injection' from the left;
- Dye (material line) stretched exponentially.

Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

[movie 1]

Channel flow

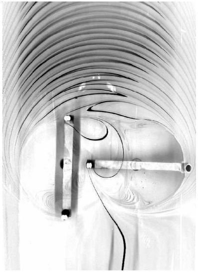


Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

[movie 2] [movie 3]

Channel flow: Injection

Injection against flow



Injection with flow

- Four-rod stirring device, with two ghost rods;
- Channel flow is upwards;
- Direction of rotation matters a lot!
- This is because it changes the injection point.
- Flow breaks symmetry.

Goals:

- Connect features to topology of rod motion: stretching rate, injection point, mixing region;
- Use topology to optimise stirring devices.

Mathematical description

Periodic stirring protocols in two dimensions can be described by a homeomorphism $\varphi : \mathcal{S} \longrightarrow \mathcal{S}$, where $\mathcal{S}$ is a compact orientable surface.

For instance, in the previous slides,

- φ describes the mapping of fluid elements after one full period of stirring, obtained from solving the Stokes equation;
- $\mathcal{S}$ is the disc with holes in it, corresponding to the stirring rods.

Task: Categorise all possible φ.

φ and ψ are isotopic if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

Thurston–Nielsen classification theorem

φ is isotopic to a homeomorphism φ', where φ' is in one of the following three categories:

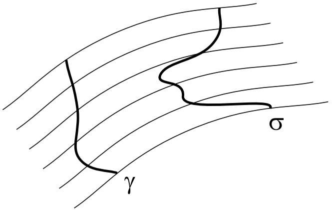
1. finite-order: for some integer $k > 0$, ${\varphi'}^k \simeq$ identity;
2. reducible: φ' leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;
3. pseudo-Anosov: φ' leaves invariant a pair of transverse measured singular foliations, $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$, such that $\varphi'(\mathcal{F}^{\mathrm{u}}, \mu^{\mathrm{u}}) = (\mathcal{F}^{\mathrm{u}}, \lambda\, \mu^{\mathrm{u}})$ and $\varphi'(\mathcal{F}^{\mathrm{s}}, \mu^{\mathrm{s}}) = (\mathcal{F}^{\mathrm{s}}, \lambda^{-1} \mu^{\mathrm{s}})$, for dilatation $\lambda \in \mathbb{R}_+$, $\lambda > 1$.

The three categories characterise the isotopy class of φ.

Number 3 is the one we want for good mixing

What's a foliation?

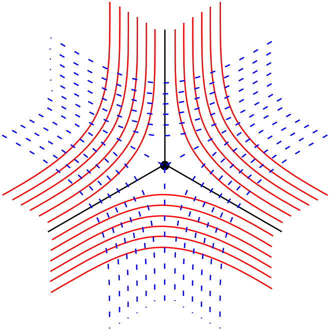
- A pseudo-Anosov (pA) homeomorphism stretches and folds a bundle of lines (leaves) after each application.
- This bundle is called the unstable foliation, $\mathcal{F}^u$.
- Arcs are measured by 'counting' the number of leaves crossed.
- Two arcs transverse to a foliation $\mathcal{F}$, with the same transverse measure.



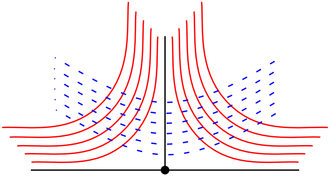
- If we iterate φ, the transverse measure of these arcs increases by a factor λ.

A singular foliation

The ‘pseudo’ in pseudo-Anosov refers to the fact that the foliations can have a finite number of pronged singularities.



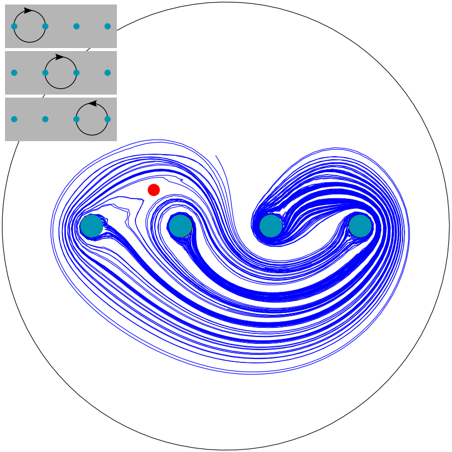
3-pronged singularity



Boundary singularity

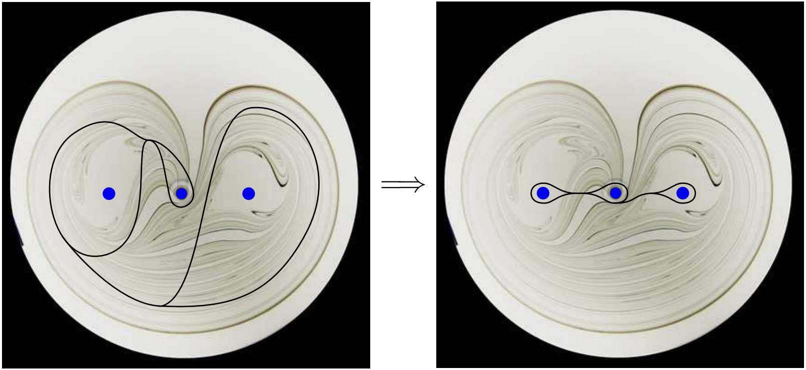
But do these things exist?

Visualising a singular foliation



- A four-rod stirring protocol;
- Material lines trace out leaves of the unstable foliation;
- One 3-pronged singularity.
- One injection point (top): corresponds to boundary singularity;

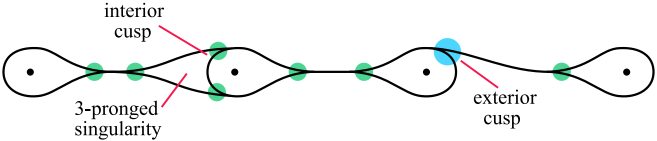
Train tracks

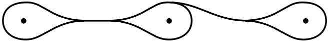


Thurston introduced train tracks as a way of characterising the measured foliation. The name stems from the ‘cusps’ that look like train switches.

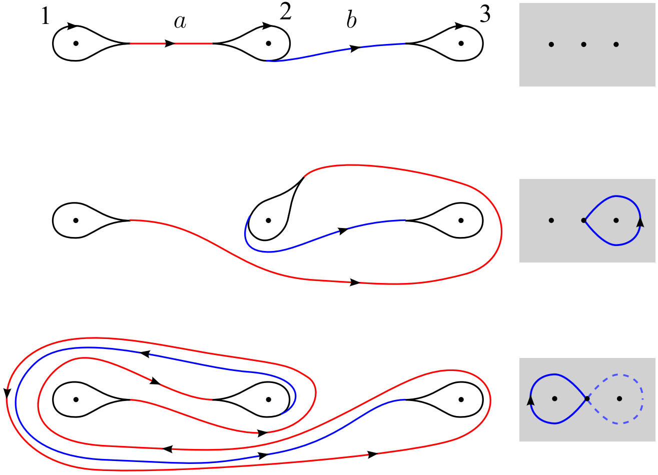
What are train tracks good for?

- They tell us the possible types of measured foliations.
- Exterior cusps correspond to boundary singularities.

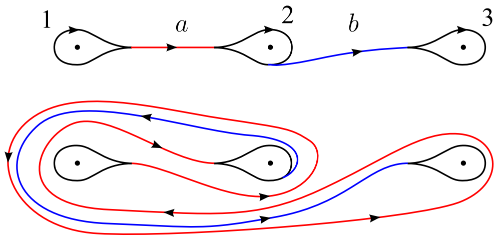


- These exterior cusp are the injection points.
- For three rods, only one type! 
- The stirring protocol gives the train track map.
- Stokes flow reproduces these features remarkably well.

Train track map for figure-eight



Train track map: symbolic form



$$a \mapsto a\,\bar{2}\,\bar{a}\,\bar{1}\,a\,b\,\bar{3}\,\bar{b}\,\bar{a}\,1\,a\,, \qquad b \mapsto \bar{2}\,\bar{a}\,\bar{1}\,a\,b$$

Easy to show that this map is efficient: under repeated iteration, cancellations of the type $a\,\bar{a}$ or $b\,\bar{b}$ never occur.

There are algorithms, such as Bestvina & Handel (1992), to find efficient train tracks. (Toby Hall has an implementation in C++.)

Topological Entropy

As the TT map is iterated, the number of symbols grows exponentially, at a rate given by the topological entropy, $\log \lambda$. This is a lower bound on the minimal length of a material line caught on the rods.

Find from the TT map by Abelianising: count the number of occurences of a and b, and write as matrix:

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The largest eigenvalue of the matrix is $\lambda = 1 + \sqrt{2} \simeq 2.41$. Hence, asymptotically, the length of the 'blob' is multiplied by 2.41 for each full stirring period.

Index formulas

To classify the possible train tracks for n rods, we use two index formulas: these are standard and relate singularities to topological invariants, such as the Euler characteristic, χ, of a surface.

Start with a sphere, which has $\chi = 2$. Each rod decreases χ by 1 (Euler–Poincaré formula), and the outer boundary counts as a rod. Thus, for our stirring device with n rods, we have
$\chi = 2 - (n + 1) = 1 - n$.

Now for the first index theorem: the maximum number of singularities in the foliation is $-2\chi = 2(n - 1)$.

n	max singularities	max bulk singularities
3	4	0
4	6	1
5	8	2

Second index formula

$$\sum_{\text{singularities}} \{2 - \#\text{prongs}\} = 2\chi(\text{sphere}) = 4$$

where #prongs is the number of prongs in each singularity (1-prong, 3-prong, etc).

Thus, each type of singularity gets a weight:

$\#$prongs	$\{2 - \#\text{prongs}\}$	
1	1	only case with $\{2 - \#\text{prongs}\} > 0$
2	0	hyperbolic point ()
3	-1	
4	-2	

Counting singularities: 3 rods

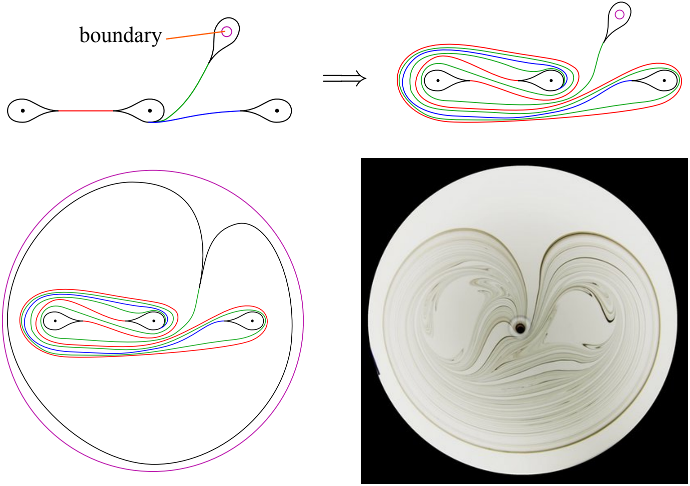
Each rod has a 1-prong singularity (). Hence, for 3 rods,

$$3 \cdot 1 + N = 4 \quad \Longrightarrow \quad N = 1\,.$$

A 1-prong is the only way to have $\{2 - \#\text{prongs}\} > 0$, hence there must be another one-prong! This corresponds to a boundary singularity.

Our first index theorem says that there can be no other singularities in the foliation.

The Boundary Singularity



Kidney-shaped mixing regions are thus ubiquitous for 3 rods.

Counting singularities: 4 rods

For 4 rods,

$$4 \cdot 1 + N = 4 \quad \Longrightarrow \quad N = 0\,.$$

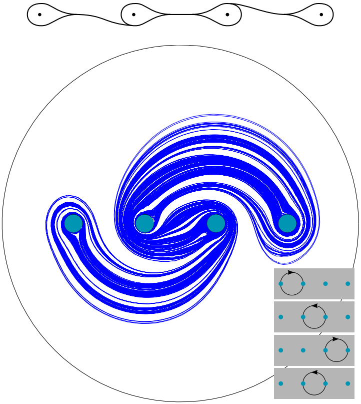
Since every boundary component must have a singularity (part of the TN theorem), two cases:

1. A 2-prong singularity on the boundary ($N = 0$), or
2. A 1-prong on the boundary and a 3-prong in the bulk ($N = 1 - 1 = 0$).

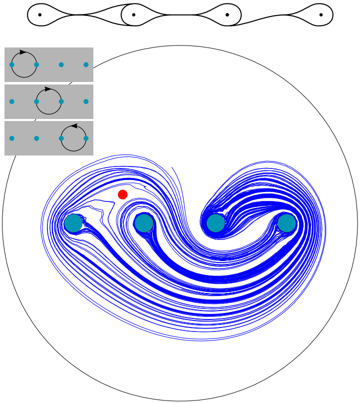
Again, our first index formula says that we are limited to one bulk singularity.

$\Longrightarrow$ Two types of train tracks for $n = 4$!

Two types of stirring protocols for 4 rods

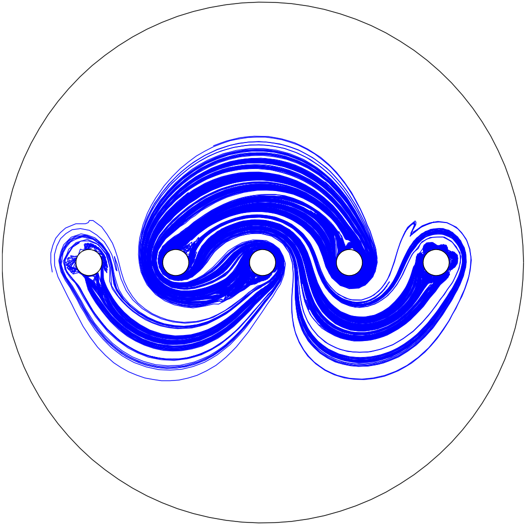


2 injection points
Cannot be on same side

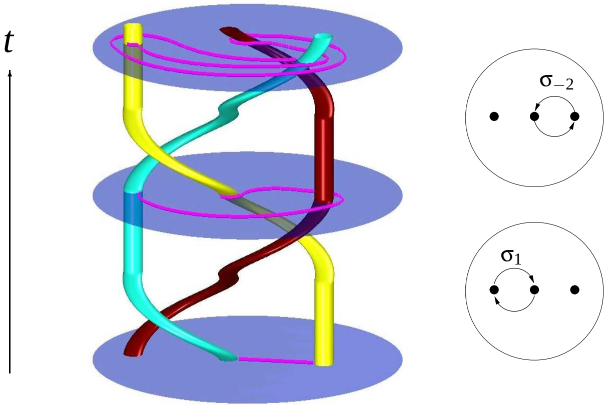


1 injection point
1 3-prong singularity

Five Rods, 3 Injection Points



The Connection with Braids



[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]
Picture from [E. Gouillart, M. D. Finn, and J.-L.Thiffeault, *Phys. Rev. E* **73**, 036311 (2006)]

Optimal Braids

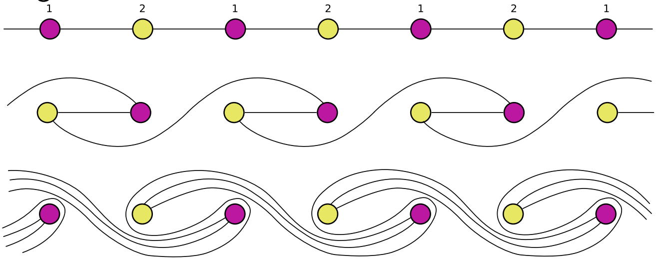
- The stretching of material lines is bounded from below by the braid's topological entropy.
- D'Alessandro et al. (1999) showed that $\sigma_1 \sigma_2^{-1}$ is optimal for 3 rods.
- This means that it has the most entropy per generator, in this case equal to $\log \phi$, where ϕ is the Golden Ratio.
- For $n > 3$ rods, all we have are conjectures (Thiffeault & Finn, 2006; Moussafir, 2006):
  - For $n = 4$, the optimal braid is $\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}$, also with entropy per generator $\log \phi$;
  - For $n > 4$, the entropy per generator is always less than $\log \phi$.

The Right Optimality?

- Entropy per generator is interesting, but does not map to physical situations very well:
- Simple rod motions can correspond to too many generators.
- In practice, need generators that are more naturally suited to the mechanical constraints.
- Another problem is that in practical sitations it is desirable to move many rods at once.
- Energy constraint not as important as speed and simplicity.
- $\sigma_1 \sigma_2^{-1}$ not so easy to realise mechanically, though see Binder & Cox (2007) and Kobayashi & Umeda (2006).

Solution: Rods in a Circle

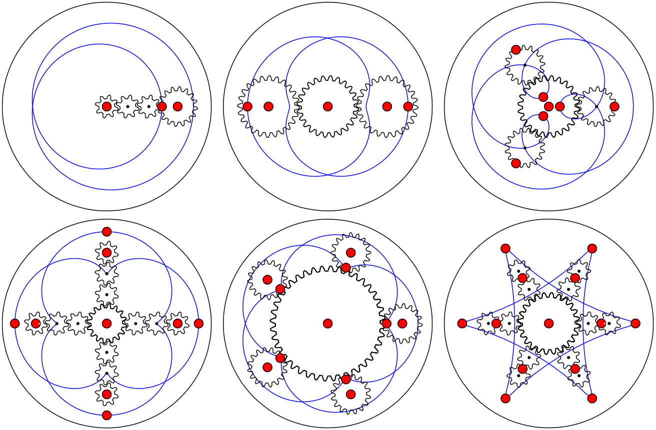
- A mixer design consisting of an even number of rods in a circle.
- Move all the rods such that they execute $\sigma_1\,\sigma_2^{-1}$ with their neighbor.



- The entropy per 'switch' is $\log\chi$, where $\chi = 1 + \sqrt{2}$ is the Silver Ratio!
- This is optimal for a periodic lattice of two rods (Follows from D'Alessandro et al. (1999)).

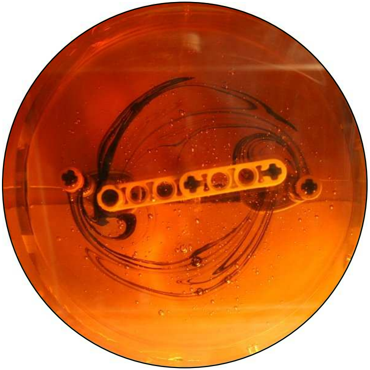
Silver Mixers!

- Even better: the designs with entropy given by the silver ratio can be realised with simple gears.
- All the rods move at once: very efficient.



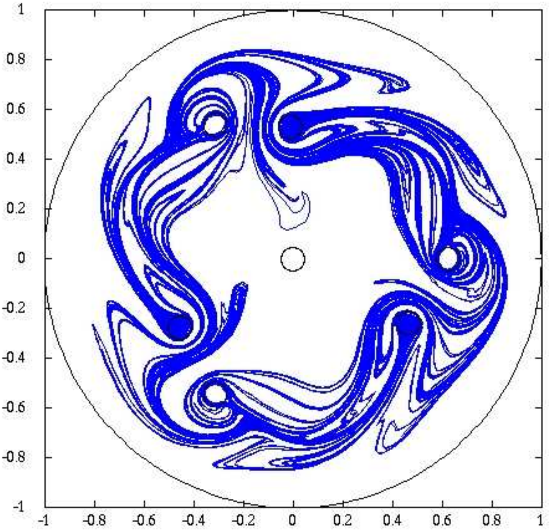
[movie 4]

Four Rods



[movie 5] [movie 6] [movie 7]

Six Rods



[movie 8]

Conclusions

- Having rods undergo ‘braiding’ motion guarantees a minimal amound of entropy (stretching of material lines).
- Topology also predicts injection into the mixing region, important for open flows.
- Classify all rod motions according to their topological properties.
- More generally: Periodic orbits! (ghost rods and folding)
- We have an optimal design (silver mixers), but more can be done.
- Need to also optimise other mixing measures, such as variance decay rate.
- Three dimensions! (microfluidics)

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