The cat's cradle, stirring, and topological complexity

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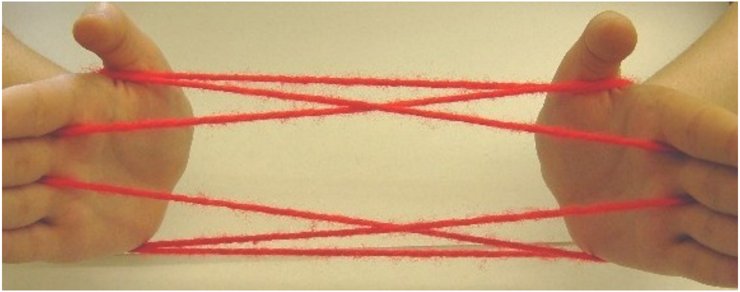
UW–Madison Math Club, 7 October 2013

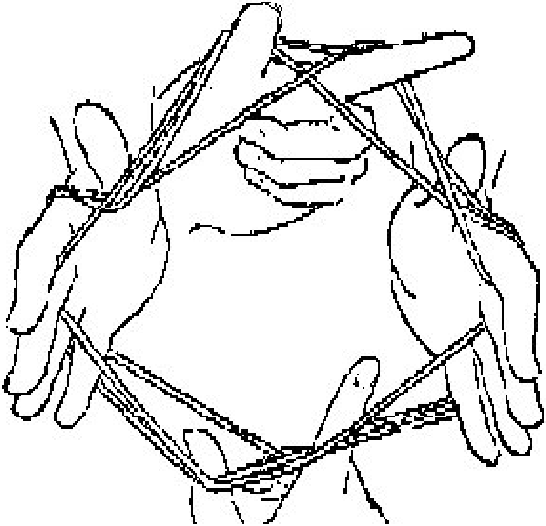
Based on a paper in Dynamical Systems Magazine, April 2009.

The cat's cradle

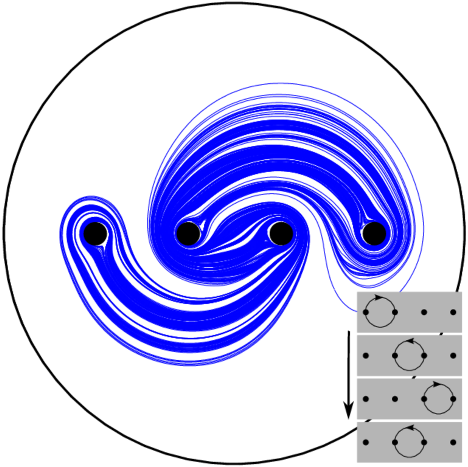


(Hands by Sarah Tumasz)

The cat's cradle: How to

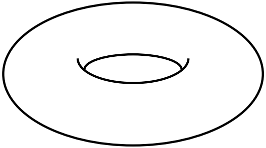


Stirring a fluid

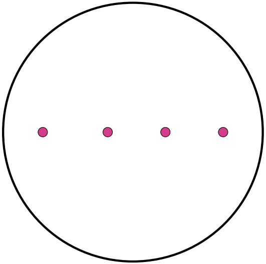


[movie 1] [movie 2]

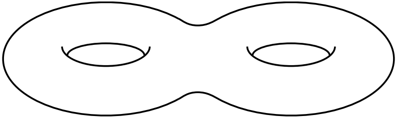
Surfaces: Holes and handles



torus (genus $= 1$)



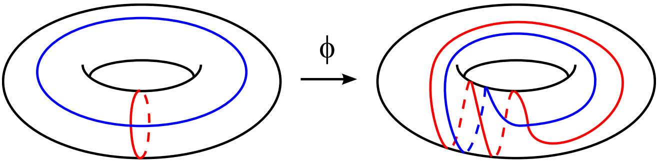
disk with 4 holes



2-torus (genus $= 2$)

Mappings of surfaces

A continuous, invertible mapping ϕ and its action on two closed loops:



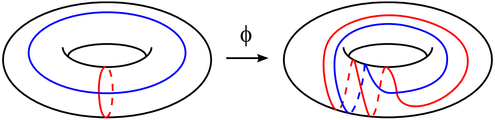
ϕ is called a homeomorphism.

In topology, only care about objects up to continuous deformation (homotopy) — equivalence classes of loops.

Torus: Types of mappings

The topological types of mappings define the mapping class group of a surface. For the torus, all that matters is what a mapping ϕ does to loops.

$\Longrightarrow$ count how many times loops wrap around periodic directions.



Under the action of ϕ:

red: $\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}2\\1\end{pmatrix}$ blue: $\begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\1\end{pmatrix}$

This is a linear transformation.

Torus: Action on loops

Hence, the action on fundamental loops

$$\text{red: } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{blue: } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

can be written as a matrix equation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

encapsulates everything we need to know (topologically) about the mapping ϕ.

Torus: Mapping class group

In mathematical language, we say that the the mapping class group of the torus is isomorphic to matrices:

$$\mathrm{MCG}(\text{torus}) \simeq \mathrm{SL}_2(\mathbb{Z})$$

with $\mathrm{SL}_2(\mathbb{Z})$ the group of invertible two-by-two matrices with determinant 1.

(A positive determinant guarantees orientability, and unit determinant means that the matrices can be inverted over the integers.)

How can we understand the types of possible behavior of different elements of the mapping class group?

Classification of mappings

Consider a general element of MCG(torus), represented here as an integer matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad - bc = 1$$

One way to classify mapping classes is to examine the eigenvalues of M.

Since $\det M = \lambda_1 \lambda_2 = 1$, M has two eigenvalues

$$\lambda_1 = \lambda \text{ and } \lambda_2 = \lambda^{-1}.$$

Without loss of generality, we assume $|\lambda| \geq 1$. (λ could be complex.)

Characteristic polynomial

Recall the characteristic polynomial

$$
\begin{aligned}
p(x) = \det(M - x\,I) = \begin{vmatrix} a - x & b \\ c & d - x \end{vmatrix} &= (a - x)(d - x) - bc \\
&= x^2 - xd - ax + ad - bc \\
&= x^2 - (a + d)x + 1 \\
&= x^2 - \tau\,x + 1
\end{aligned}
$$

where $\tau = a + d$ is the trace of M.

Cayley–Hamilton theorem

The Cayley–Hamilton theorem states that any matrix is a zero of its characteristic equation:

$$p(M) = M^2 - \tau\, M + I = 0 \quad \Longleftrightarrow \quad M^2 = \tau\, M - I$$

We can write M^2 in terms of M!

Example:

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \Longrightarrow \quad \tau = 2 + 1 = 3,$$

$$M^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, \qquad 3M - I = 3\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

Classification, part 1

$$M^2 = \tau\, M - I$$

$\tau = a + d$ is the sum of two integers, so the smallest values $|\tau|$ can take are 0 and ± 1.

$\tau = 0$: then $M^2 = -I$, so $M^4 = (-I)^2 = I$.

$$\boxed{M^4 = I \qquad \text{for } \tau = 0}$$

$\tau = \pm 1$: then $M^2 = \pm M - I$, so

$$M^3 = M(M^2) = M(\pm M - I) = \pm M^2 - M = \pm(\pm M - I) - M = \mp I$$

$$\boxed{M^6 = I \qquad \text{for } \tau = \pm 1}$$

Finite-order mappings

Such mappings are called finite order: the loops eventually return to their initial winding.

We can combine the previous two statements as

$$\boxed{M^{12} = I \qquad |\tau| < 2}$$

(12 is the least common multiple of 4 and 6.)

Classification, part 2

Now consider $|\tau| > 2$.

We solve $p(x) = 0$, and find the largest eigenvalue

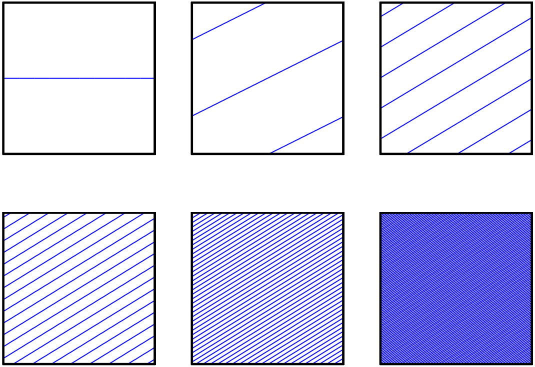
$$\lambda = \operatorname{sign}\tau \times \frac{1}{2}\left(|\tau| + \sqrt{\tau^2 - 4}\right)$$

where λ is real and $|\lambda| > 1$ (i.e., strict).

The mapping classes with $|\lambda| > 1$ are called Anosov.

Anosov maps

Representing the torus as a doubly-periodic interval $[0, 1] \times [0.1]$, first few iterates on a loop:



Asymptotically, the length of the loop is multiplied by $|\lambda|$ each time $\Rightarrow$ exponential growth.

Arnold's Cat map

For our example

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

(shown on previous slide), we have

$$\lambda = \tfrac{1}{2}(3 + \sqrt{5}) = \varphi^2 \simeq 2.618\ldots$$

where φ is the Golden ratio.

For historical reasons, this particular linear map is called Arnold's cat map, after V. I. Arnold.

Anosov maps exhibit the purest form of chaotic behavior. They can only exist on a torus.

Thurston–Nielsen classification theorem

More complex surfaces than the torus, such as the 2-torus or disks with holes, require a powerful theorem:

Theorem (Thurston–Nielsen classification theorem)

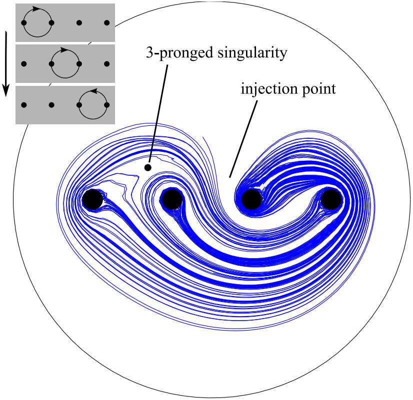
Let ϕ by a homeomorphism of a surface M. The ϕ is isotopic to ψ, where ψ is one of three types:

1. *finite-order*
2. *pseudo-Anosov*
3. *reducible*

(For the experts: this is only for hyperbolic surfaces.)

pseudo-Anosovs

pseudo-Anosovs are much more complex (and interesting!) than Anosovs, and are still the subject of active story. The "pseudo" comes from the presence of singularities.



Summary

- Mappings of surfaces can be understood from their action on curves.
- Physically, these arise for instance in fluid dynamics when stirring with rods.
- In that case, the best stirring methods correspond to pseudo-Anosov (pA) mappings.
- Many open questions about pAs:
  - Spectrum of values of λ;
  - How to construct for characteristic polynomial;
  - Connection to number theory?
- See article at arxiv.org/abs/0904.0778 for more details.