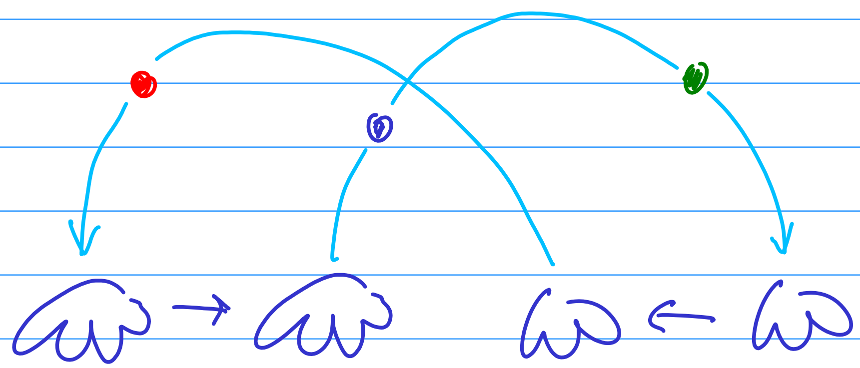
Mathematics of juggling

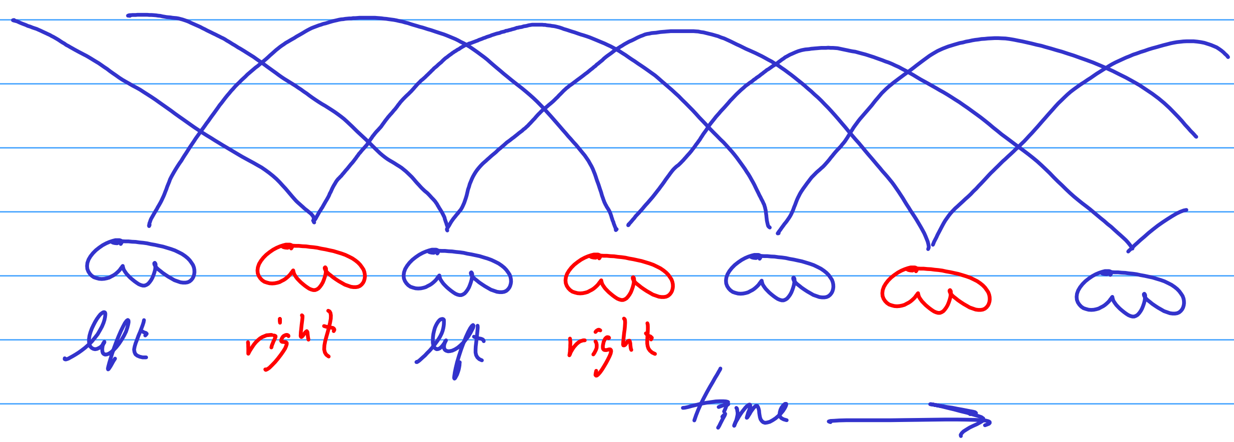
[Source: mostly the book by Polster]

[Show pattern "3" with JuggleKrazy]

"3-ball cascade"



Jugglers have invented a clever notation: "site swap"



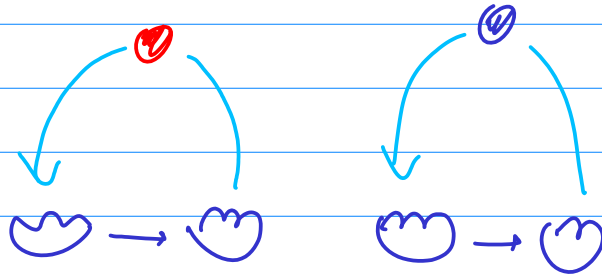
Just record how much time (in "beats") a ball is in the air. Each hand throws a ball at each beat.

The "airtime" here is 3 beats for each ball.

So write this sequence as $\ldots 3, 3, 3, 3, 3, \ldots$

or just "3".

An even number means the same hand catches the ball:

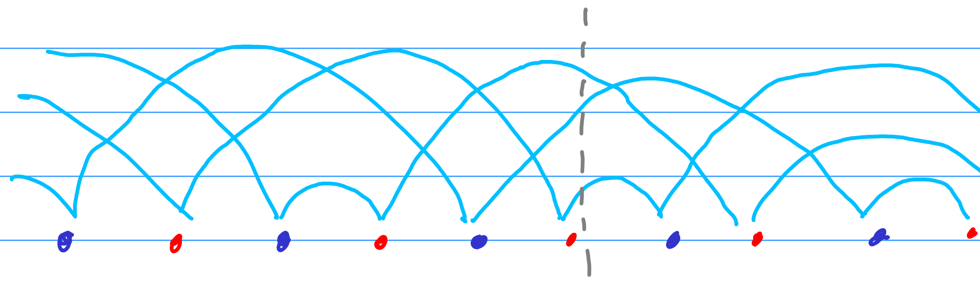


"n-ball fountain"

$\ldots n, n, n \ldots$

(n even)

441:

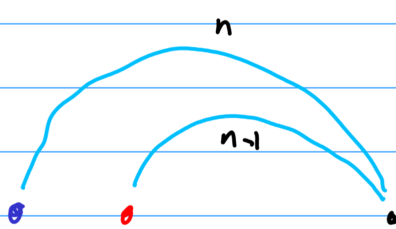


Note that 441 requires 3 balls. We can see this by drawing a vertical plane, crossed by 3 balls.

What's the rule? The number of balls is the average of the pattern: $(4+4+1)/3 = 3$.

This hints that not all sequences can be juggled.

Also, $n\ (n-1)$ is not possible,



same hand gets two balls

Let's formulate this mathematically.

$$j : \mathbb{Z} \to \mathbb{N}^0 \quad \text{is a juggling sequence}$$

$$\cdots j(-2)\, j(-1)\, j(0)\, j(1)\, j(2) \cdots$$

(0 is allowed: just hold the ball for 2 beats.)

but it has to satisfy the condition that the juggling diagram "works out".

Theorem: j is a juggling function iff

$$j_+ : \mathbb{Z} \to \mathbb{Z} \; : \; i \mapsto i + j(i)$$

is a permutation of the integers.

Buhler, Eisenbud, Graham, Wright.

Theorem: The number of juggling sequences of length n with b balls is

$$\frac{1}{n} \sum_{d \mid n} \mu(n/d) \left((b+1)^d - b^d \right)$$

up to "equivalent" patterns

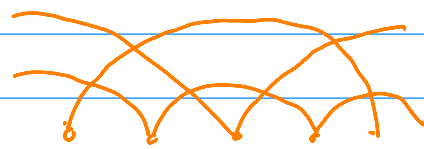
where μ is the Möbius function:

$$\mu(n) = \begin{cases} 0, & n \text{ is not square-free} \\ \pm 1, & n \text{ is square-free with even/odd number of prime factors} \end{cases}$$

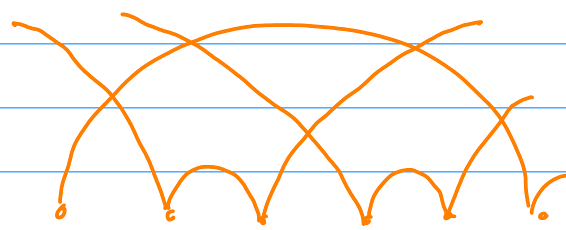
For example, with 3 balls, there is

1 pattern of length 1   3
3 patterns of length 2   42, 51, 60
12 " " " 3
42 " " " 4
156 " " " 5
⋮

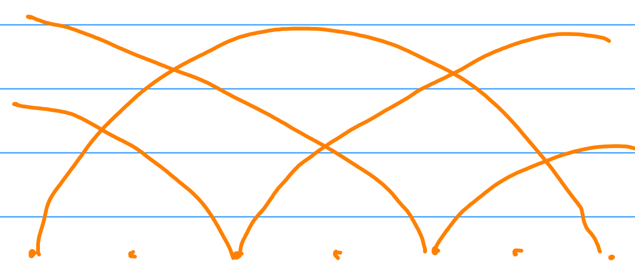
42



51



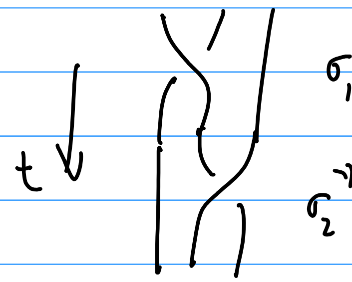
60



Juggling braids: "walk backwards" as you juggle.

The orbits of the balls trace out a braid.

For example, 3



trace out $\sigma_1\sigma_2^{-1}$

All braids can be juggled, given an infinite number of hands and balls!

But what about for 2 hands, b balls?

Macauley (2003, Harvey Mudd senior thesis) has some answers.

More: applications? relation to bell-ringing, ...