Making taffy with the Golden mean

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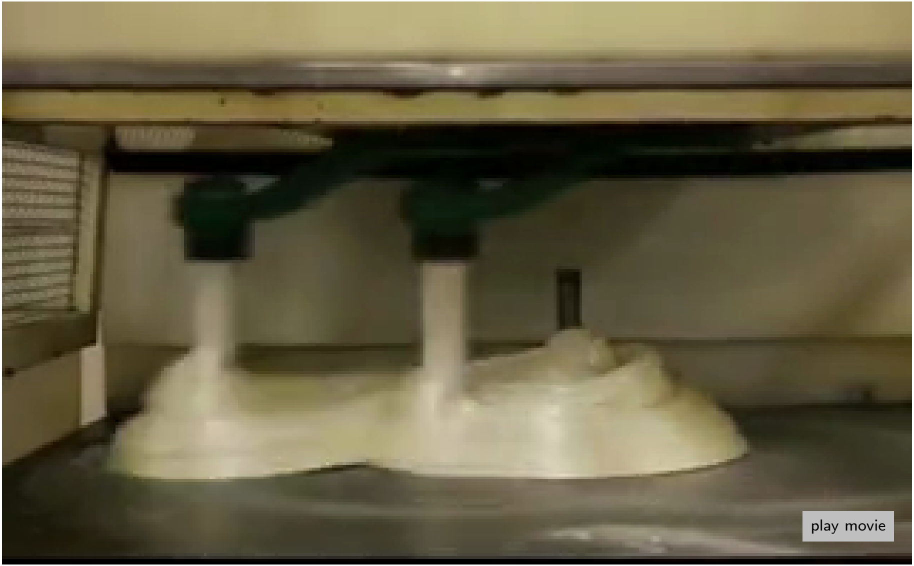
making candy by hand

play movie http://www.youtube.com/watch?v=pCLYieehzGs

taffy pullers

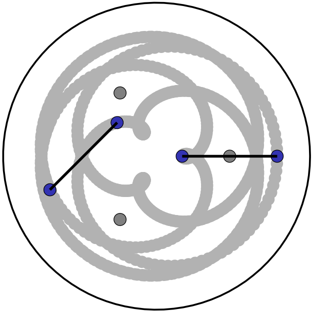
play movie play movie http://www.youtube.com/watch?v=YPP2_Zf0IVU

making candy cane

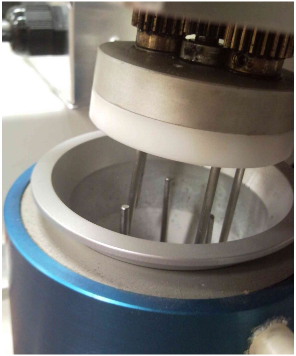


[*Wired:* This Is How You Craft 16,000 Candy Canes in a Day]

Experimental device for kneading bread dough:



play movie

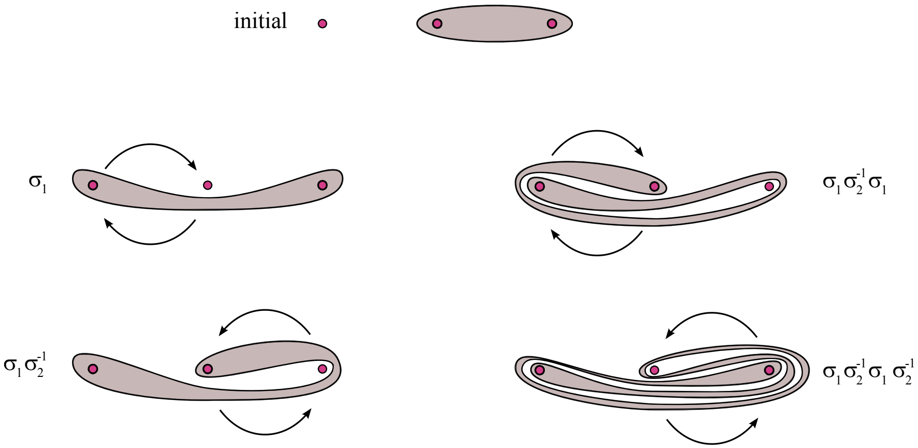


[Department of Food Science, University of Wisconsin. Photos by J-LT.]

four-pronged taffy puller

play movie http://www.youtube.com/watch?v=Y7tlHDsquVM

a simple taffy puller



number of folds

[Matlab: `demo1`]

Let's count alternating left/right folds.

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$$\#\text{folds} = 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34, \ldots$$

What is the rule?

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What is the rule?

$$\#\text{folds}_n = \#\text{folds}_{n-1} + \#\text{folds}_{n-2}$$

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This is the famous Fibonacci sequence, F_n.

how fast does the taffy grow?

It is well-known that for large n,

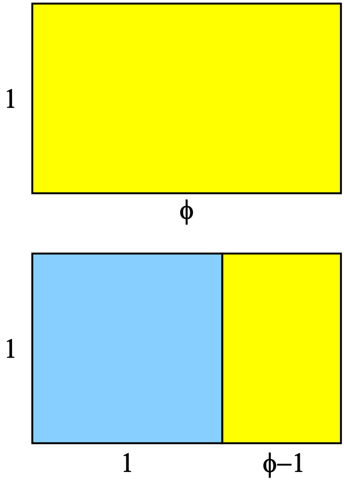
$$\frac{F_n}{F_{n-1}} \quad \to \quad \phi = \frac{1+\sqrt{5}}{2} = 1.6180\ldots$$

where ϕ is the Golden Ratio, also called the Golden Mean.

Along with π, ϕ is probably the best known number in mathematics. It seems to pop up everywhere...

So the ratio of lengths of the taffy between two successive steps is ϕ^2, where the squared is due to the left/right alternation.

the Golden Ratio, ϕ



A rectangle has the proportions of the Golden Ratio if, after taking out a square, the remaining rectangle has the same proportions as the original:

$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

[Matlab: `demo2`]

Now let's swap our prongs twice each time.

number of folds

We get for the number of left/right folds

$$\#\text{folds} = 1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ 408\ldots$$

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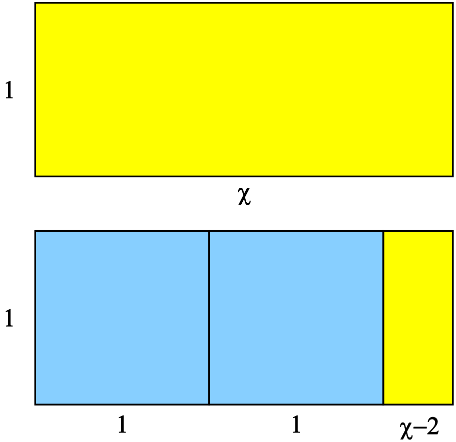
For large n,

$$\frac{\#\text{folds}_n}{\#\text{folds}_{n-1}} \quad \to \quad \chi = 1 + \sqrt{2} = 2.4142\ldots$$

where χ is the Silver Ratio, a much less known number.

A rectangle has the proportions of the Silver Ratio if, after taking out two squares, the remaining rectangle has the same proportions as the original.



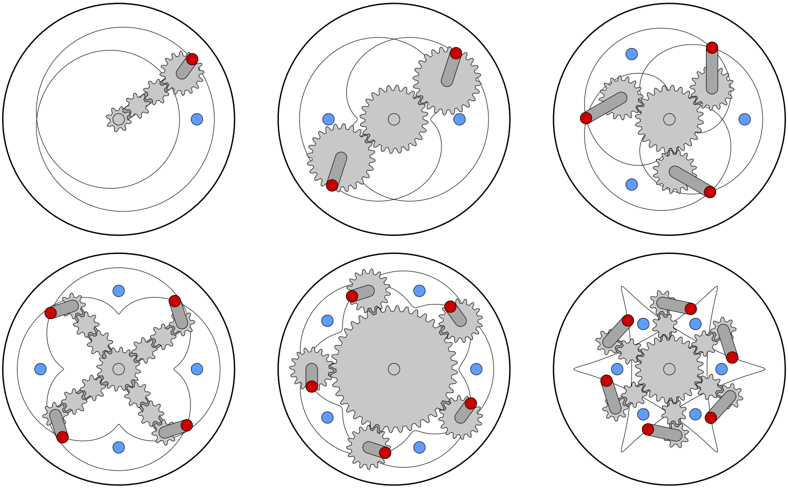
$$\frac{\chi}{1} = \frac{1}{\chi - 2}$$

the original taffy puller

The taffy puller we originally presented stretches the taffy by χ^2 at each 'period'.

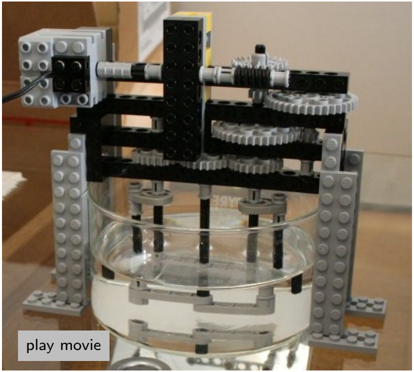
It's a special case of what we call Silver Mixers: devices that stretch by a power of the Silver Ratio.

taffy superpullers!



play movie

build it with Legos!



(picture on left appears on the cover of a math journal!)

some final thoughts

Is there a Bronze Ratio? Can we make such a taffy puller?

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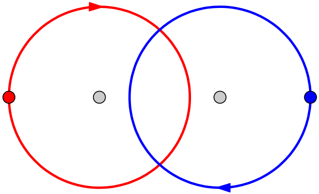
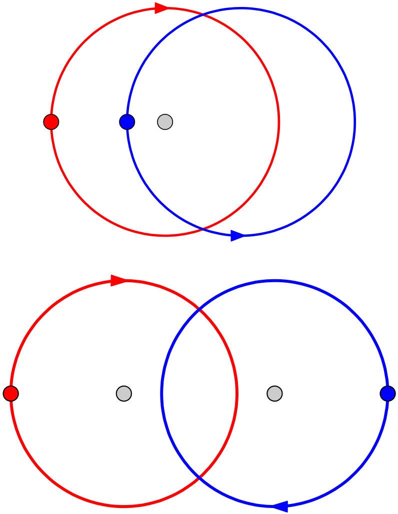
What about the taffy puller with four prongs?



Does it stretch taffy faster or slower than the three-pronged one?

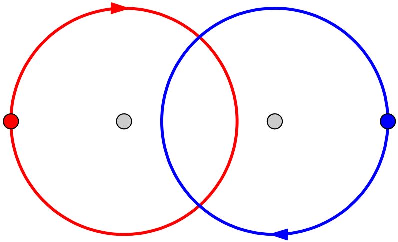
more prongs is not always better



The two 'standard' pullers have exactly the same taffy growth factor,

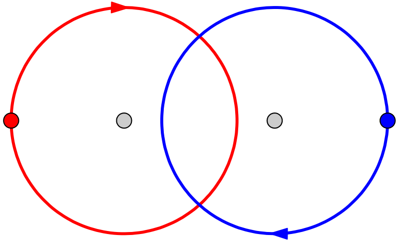
$$3 + 2\sqrt{2} \simeq 5.82843.$$

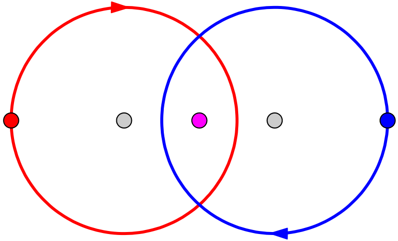
can we improve the 4-pronged puller?



It would be nice to actually gain something from adding more prongs.

can we improve the 4-pronged puller?





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Try inserting another fixed prong.

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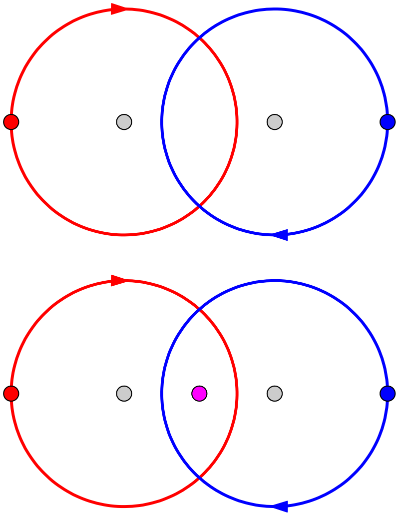


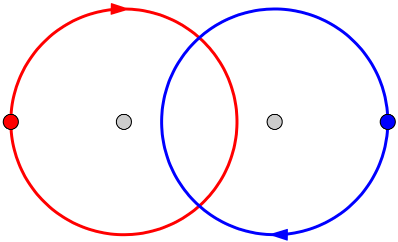
It would be nice to actually gain something from adding more prongs.

Try inserting another fixed prong.

Again, these two pullers have exactly the same taffy growth factor,

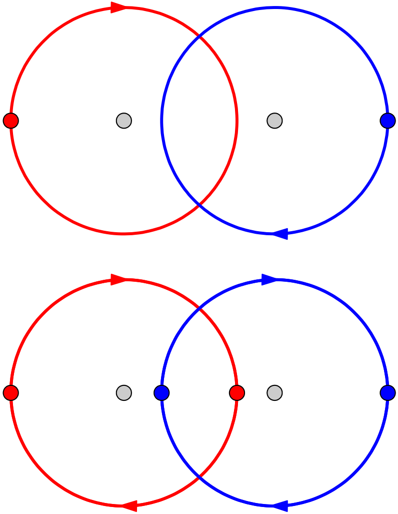
$$3 + 2\sqrt{2} \simeq 5.82843.$$

more prongs is sometimes better!



Start over!

more prongs is sometimes better!



Start over!

Use two prongs per ‘cycle.’

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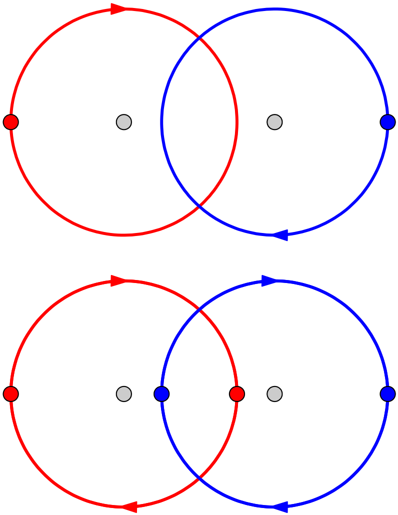


Start over!

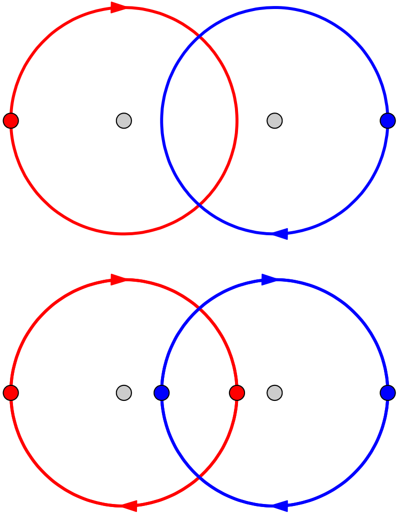
Use two prongs per ‘cycle.’

Now the taffy growth factor of the bottom puller is

$$7 + 4\sqrt{3} \simeq 13.9282,$$

which is quite a bit larger than 5.82843.

more prongs is sometimes better!



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As far as I know the bottom one has not been built...

. . . until last week!

Alex Flanagan (undergrad at UW) built one. Still need to test it out!