Stochastic field lines and braids

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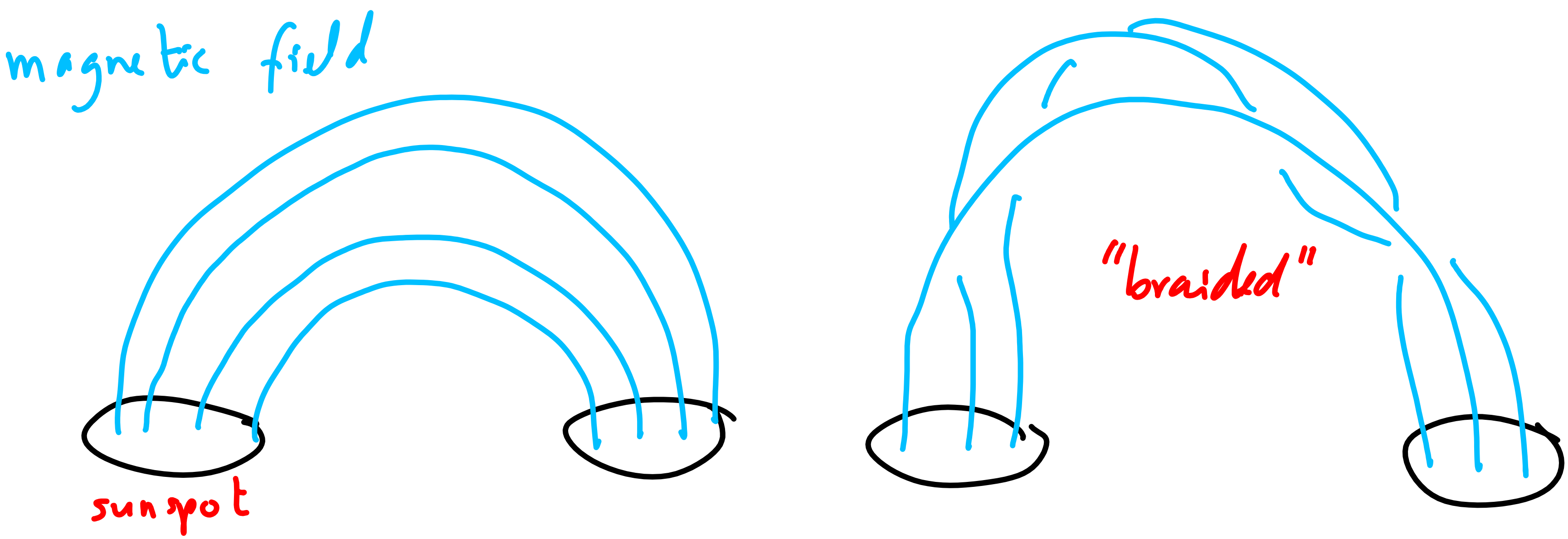
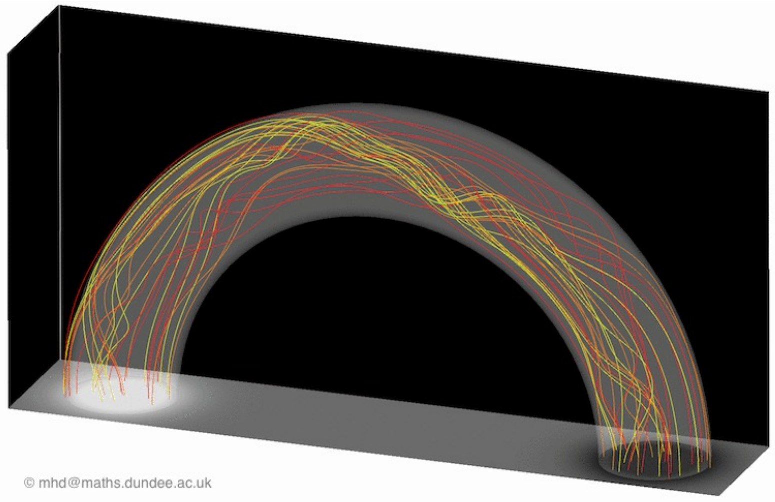
Tangled magnetic fields

The "vision" for solar flux tubes



The magnetic field lines become "braided" due to MHD frozen-in condition + turbulence

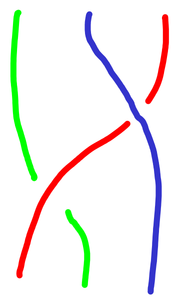
Tangled magnetic fields (cont'd)



Source: `http://www.maths.dundee.ac.uk/mhd/`

Random braids

So far studies focus on given braid: say, the pigtail braid



[Wilmot-Smith, Hornig, Yeates, ...]

But how do we generate an "appropriate" random braid?

Previous work: Berger (invariants), Sumners (random knots), Nechaev ("entangled random walks" → Cayley graphs)

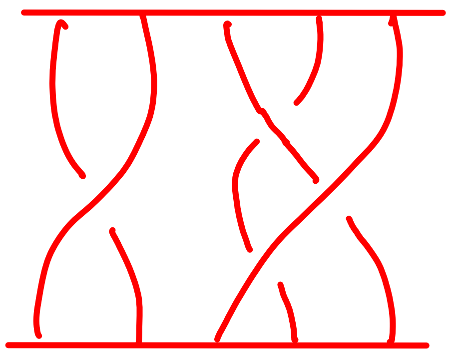
...

[more refs later]

I will summarize some earlier work and point to some difficulties.

The braid group B_n

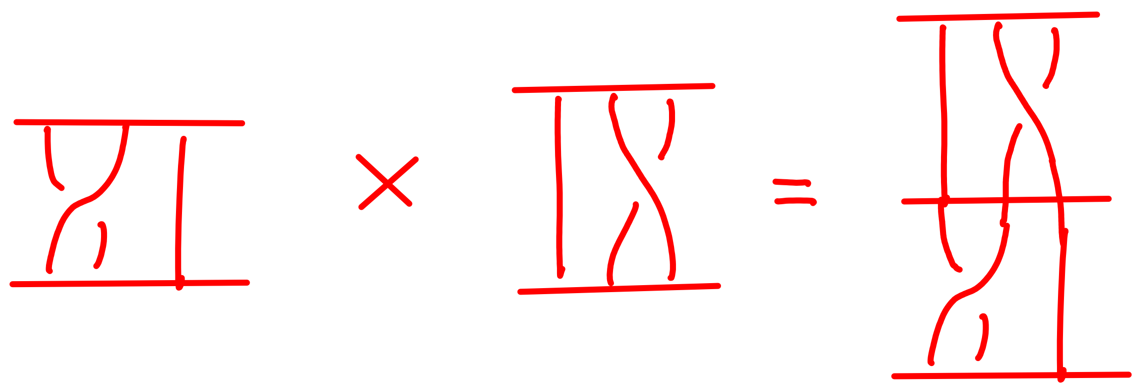
A braid is a set of n strands with fixed endpoints



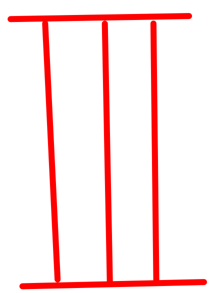
Two braids are equal if they can be "deformed" into each other, whilst holding ends fixed.

"ambient isotopy"

Braids form a group: (for fixed n)

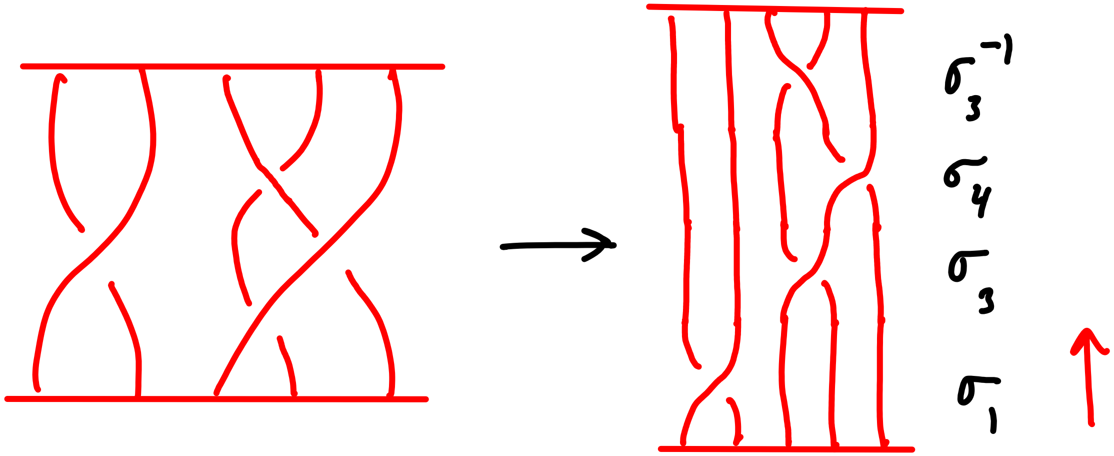


Identity:



This is associative, and for every braid there is an inverse that "disentangles" the braid.

Braid generators



This braid can be written

$$\sigma_1 \sigma_3 \sigma_4 \sigma_3^{-1}$$

$\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ are generators of B_n

They satisfy relations:

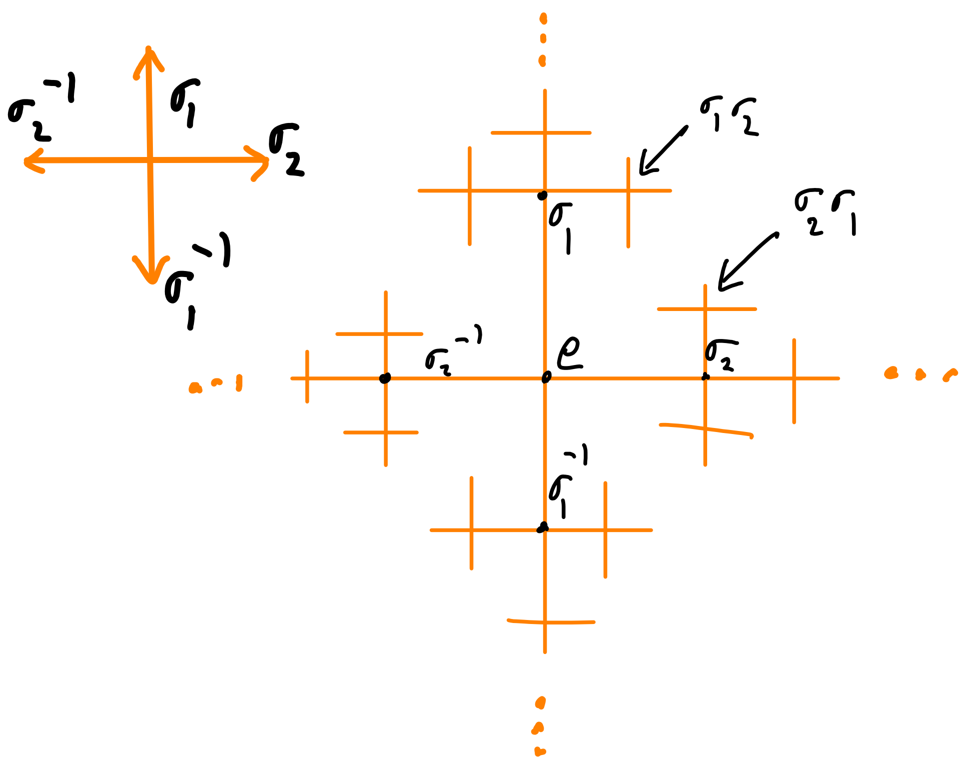
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad , \quad |i-j| > 1$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \, , \quad |i-j| = 1$$

Artin proved that these are the only relations that arise from physical braids

Cayley graph

A convenient graphical representation of groups is as a graph:



The Cayley graph for B_3 might start out like this, but there are "shortcuts" (loops) due to braid relations.

Now we can define a random walk on this graph by choosing a random direction to move to from each vertex.

γ = random word = $\sigma_1 \sigma_2 \sigma_1^{-1} \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_1 \cdots$ (say)

Lots of interesting questions! (recurrence, distance, etc...)

The simplest Cayley graph

For B_2, only one generator

The exponent gives the winding number of one strand around the other.

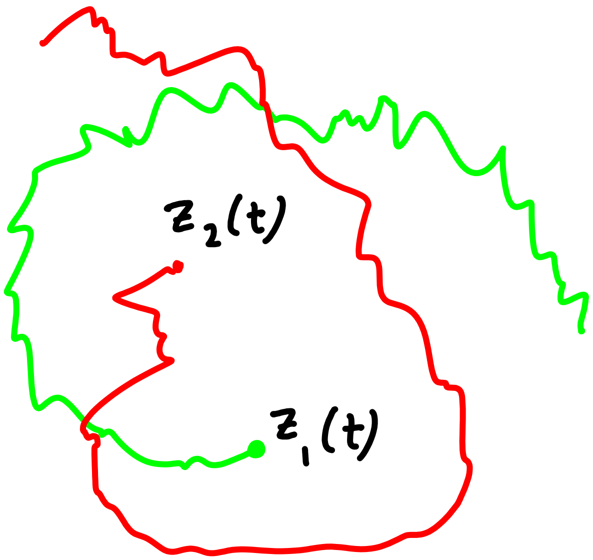
If our random walk moves to the left/right with probability $p/(1-p)$ expect net winding number m after N steps to have probability

$$P(m) = \binom{N}{k_m} p^{k_m}(1-p)^{N-k_m} \qquad k_m = \frac{1}{2}(m+N)$$

($m+N$ even!)

Mean $N(p-1/2)$, variance $Np(1-p)$ $\Rightarrow$ Gaussian for large N

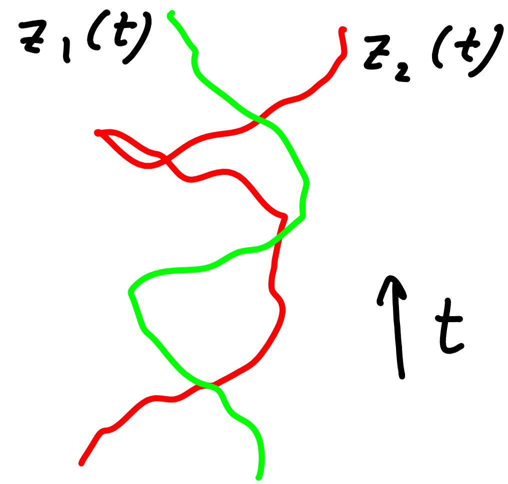
Winding number of two Brownian processes



Consider now two Brownian processes on the plane, diffusivity D. $\langle z_i^2 \rangle \sim 2Dt$

(Think of two stochastic field lines)

Picture as a braid:



What is the distribution of the winding angle, θ, after a large time t?

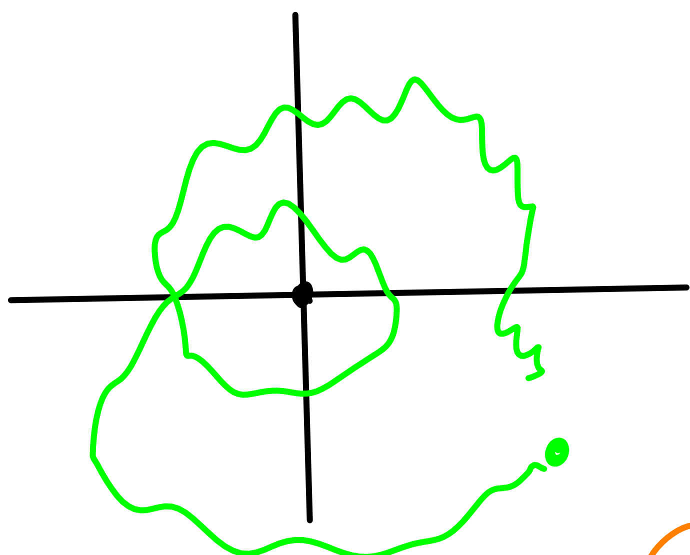
Same as Cayley graph random walk?

Winding around the origin

Let $\ell(t) = z_1(t) - z_2(t)$ $\longrightarrow$ Brownian with diffusivity 2D

$\langle \ell^2 \rangle = \langle z_1^2 \rangle + \langle z_2^2 \rangle = 2(2Dt)$

So now the question is: how many times does ℓ wind around the origin?



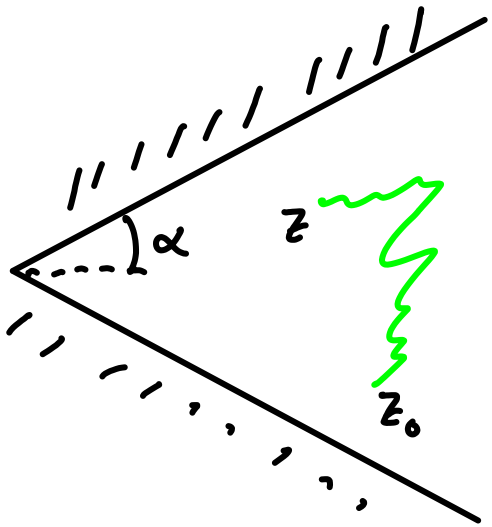
Classic result of Spitzer (1958):

$$p(x) \sim \frac{1}{\pi} \frac{1}{1+x^2}$$

$$x = 2\theta / \log(4Dt/r_0^2) \qquad \frac{Dt}{r_0^2} \gg 1$$

Cauchy-Lorentz $\longrightarrow$ not Gaussian!

How do we show this?



Brownian process ⇒ heat equation

Solve in wedge of angle 2α:

$$\frac{\partial P}{\partial t} = D\nabla^2 P \qquad P_0 = \delta(z - z_0)$$

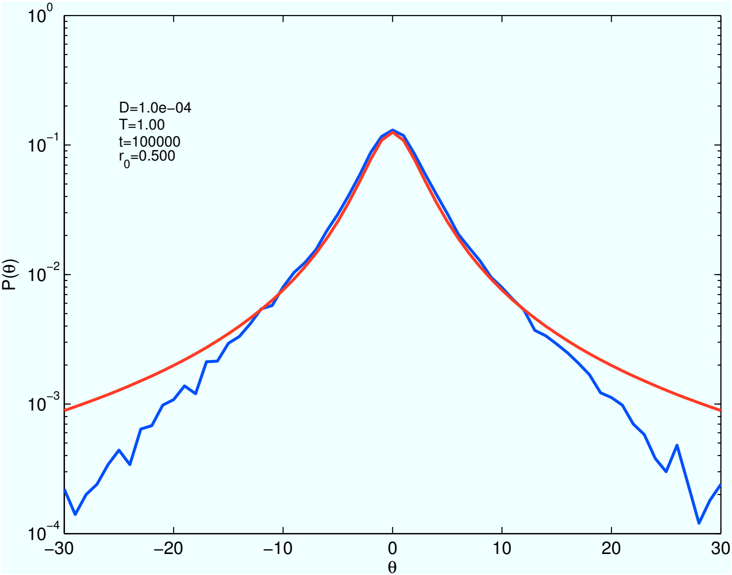
with Neumann conditions (conservation of probability)

Compute Green's function (e.g. Carslaw & Jaeger)

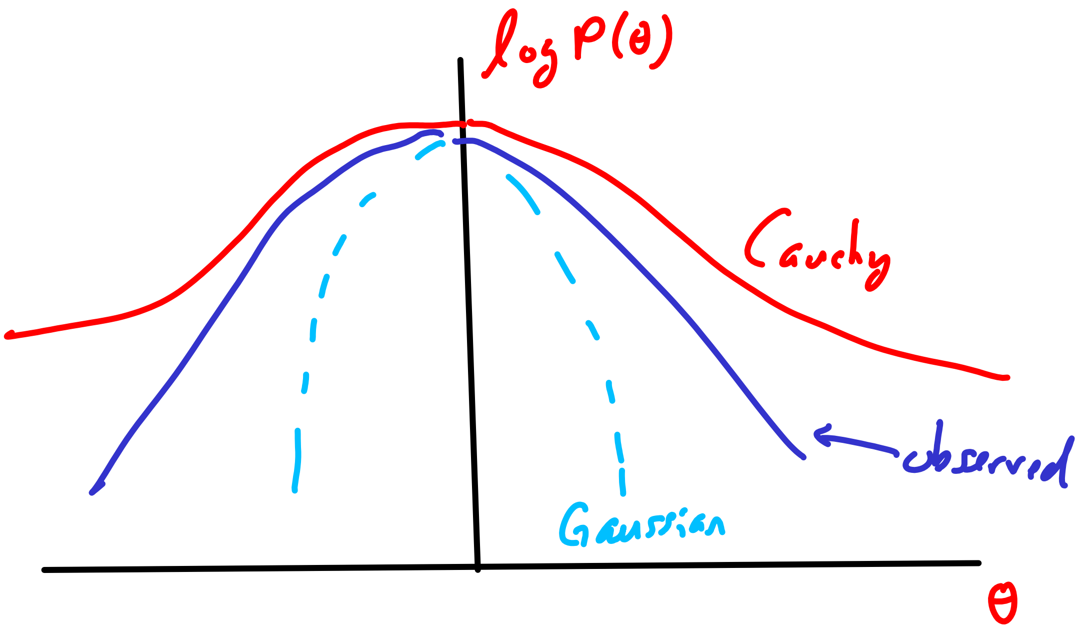
then take $\alpha \to \infty$! → multiple Riemann sheets

large – t asymptotics then give Cauchy distribution.

Numerical simulations



The problem with the tails



numerical sims.

In practice, we never see this. The large winding angles predicted by Spitzer are a symptom of the scale-free Brownian process.

→ can wind very fast around origin.

Any "regularization" (random walk, length scale, curvature-limited...) gives

$$p(x) \sim \frac{1}{2}\,\mathrm{sech}\left(\frac{\pi x}{2}\right)$$

← exponential tails (still not Gaussian)

[One way to get this: take out disk around origin]

[Pitman & Yor '86, Berger '87, Drossel & Kardar '96, Grosberg & Frisch '03]

Scaling of x with $\log t$

The scaling $x \sim 2\theta / \log(4Dt/r_0^2)$ arises because the Brownian process wanders away from the origin $\longrightarrow$ PDF stops changing

Scaling argument:

$$\frac{dr}{d\theta} = r f(\theta)$$ since r is the only length scale for $Dt \gg r_0^2$

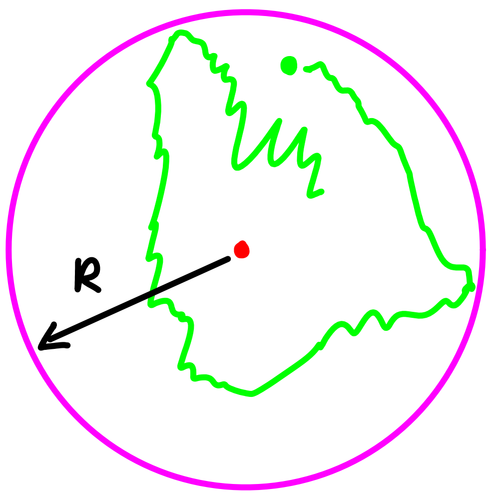
$\hookrightarrow$ indep. of θ by isotropy

So $$d\theta \sim \frac{dr}{r} \sim \frac{dt}{t} \sim d\log t$$ since $r \sim t^{1/2}$

[See Fisher et al. 1984, Drossel & Kardar 1996]

Closed domain

Now all this was for a Brownian motion on the plane.
In a confined geometry, the process "starts over" when it reflects. [Markovian]



So pieces of random walk of duration

$$\frac{R^2}{D}$$

diffusion time across disk

are uncorrelated, and each piece has a Spitzer distribution.

Convolving the $t/(R^2/D)$ distributions then gives

$$x \sim \frac{\theta}{\sqrt{t}}$$

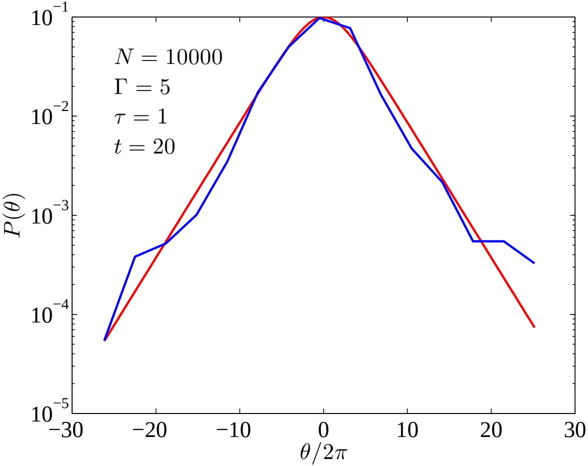
rather than $\theta/\log t$

Confinement leads to many more turns

[Drossel & Kardar 2003]

Blinking vortex simulations

Winding number for blinking vortex pair in a disk (Aref, 1984):



The red curve is the sech distribution, with fitted diffusivity.

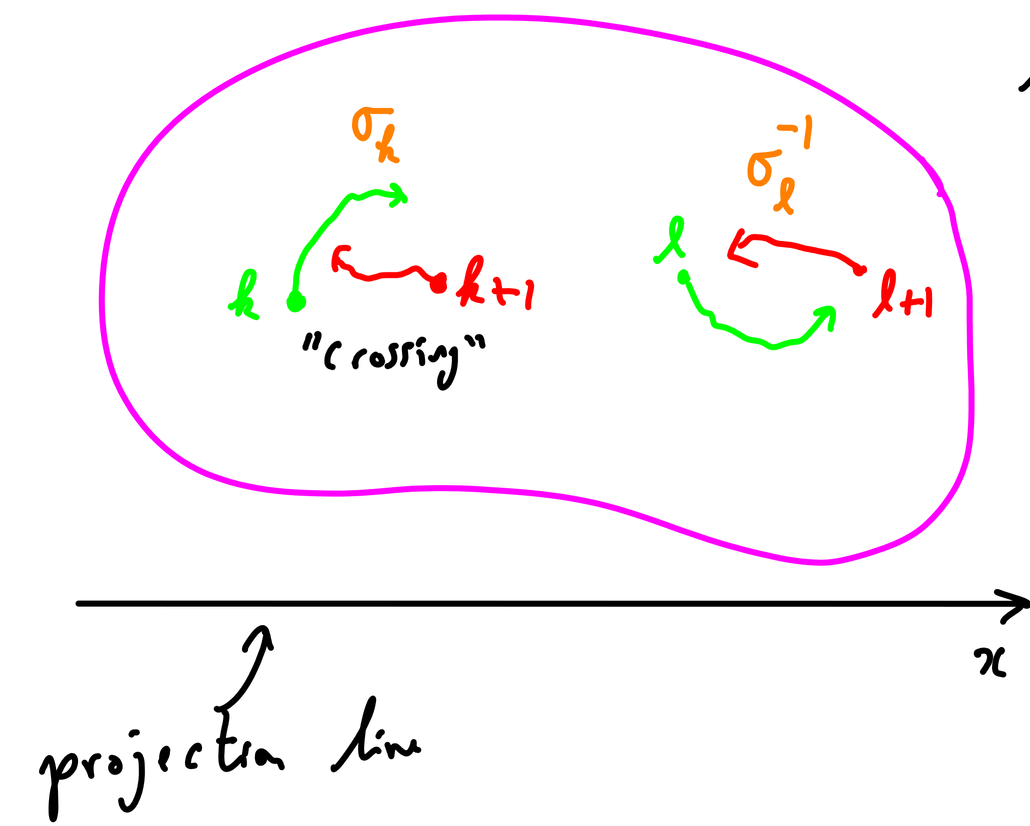
The moral so far

Maybe I've convinced you that the random walk on a Cayley graph (which suggest Gaussian winding angle) is hard to reconcile with the twisting of two random walkers. (Not sure how to do it...)

At least in our two-walker example the probability of winding in one direction was the same as the other. $P(\sigma_1) = P(\sigma_1^{-1})$

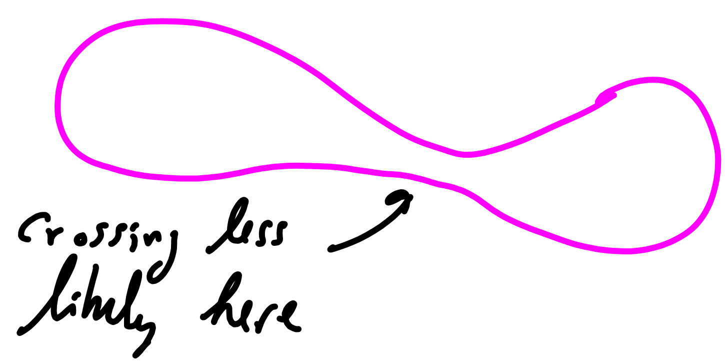
But more generally, for n random walks, are all braid generators $\{\sigma_1, \ldots, \sigma_{n-1}\}$ equally likely?

Random walkers in an arbitrary domain



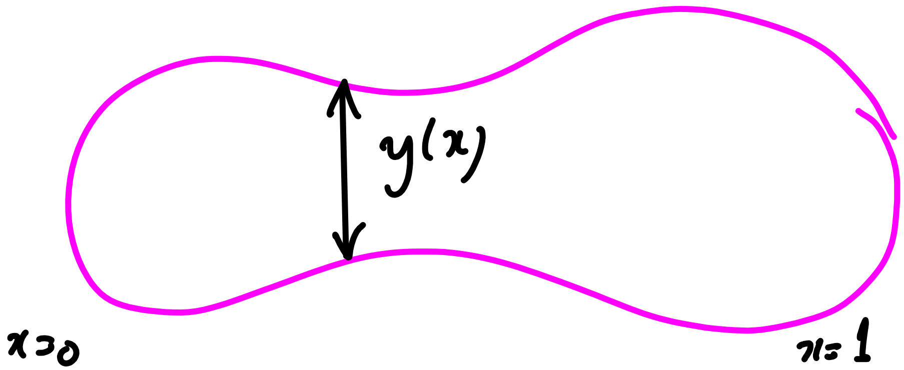
When projecting along a fixed line, each "crossing" of two walkers corresponds to a generator $\sigma_k^{\pm 1}$.

Since each walker is independent and uniformly distributed, the probability of observing a crossing depends on width:



[See J-LT 2005, 2010]

Where do crossings occur?



The probability of a walker being in $[x, x+dx]$ is

$$\rho(x)\,dx = \frac{1}{A}\,y(x)\,dx$$

area

$\left(\text{So } \int_0^1 \rho(x)\,dx = 1\right)$

The probability of a crossing occuring at x is

$$P(\text{crossing at } x) = \frac{\rho^2(x)}{\int_0^1 \rho^2(x)\,dx}$$

two particles at x is a crossing

[This is in the small step-size limit]

Which generator?

Once we have a crossing, we can ask which generator it corresponds to.

This depends on the ordering of the particles.

Find: [See Sarah Tumasz's thesis]

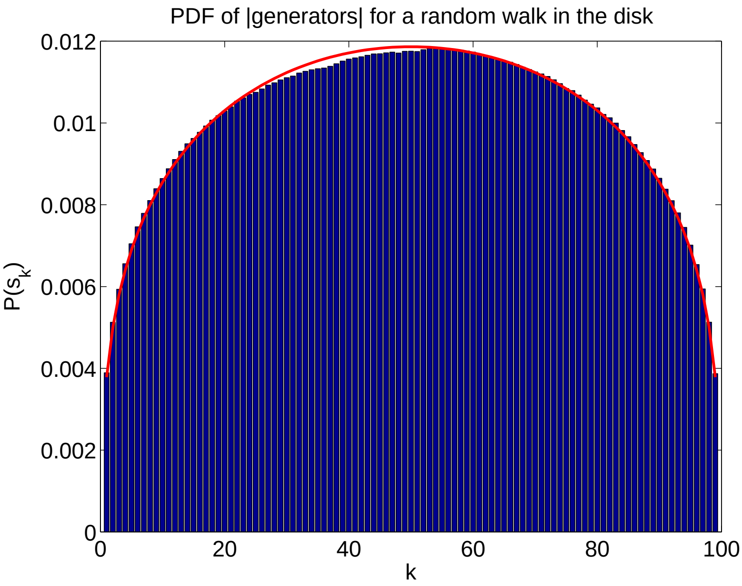
$$P(\sigma_k^{\pm 1}) = \binom{n-2}{k-1} \int_0^1 \underbrace{\rho^2(x)}_{\text{crossing at } x} \underbrace{P^{k-1}(x'<x)}_{\text{particles to the left of } k} \underbrace{P^{n-k-1}(x'>x)}_{\text{particles to the right of } k} dx$$

For a square domain, $\rho(x) = 1$ and

$$P(\sigma_k^{\pm 1}) = \frac{1}{n-1}$$

All generators projected along the x-axis occur with equal probability.

Distribution of generators in a disk



Conclusions

- Can generate braids by 'randomly picking generators' (random walk on Cayley graph), but not clear what physical process that corresponds to;
- Brownian motions have Cauchy-distributed winding angles;
- Random walks have sech-distributed winding angles, as does a simple chaotic flow;
- The braids created by random walks depend on the shape of the domain!
- Topological entropy of random braids?

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