Topological chaos

Jean-Luc Thiffeault

Department of Mathematics
University of Wisconsin – Madison

Chaos and Complex Systems Seminar
Physics Dept., University of Wisconsin, Madison
4 October 2011

The Taffy Puller

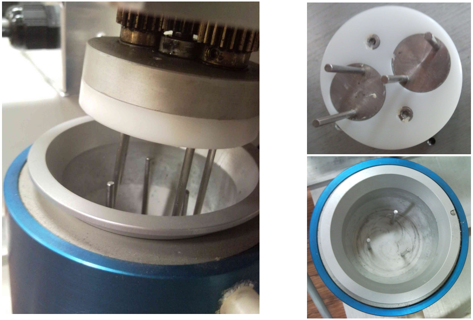
This may not look like it has much to do with stirring, but notice how the taffy is stretched and folded exponentially.

Often the hydrodynamics are less important than the topological nature of the rod motion.

[movie 1]

The mixograph

Experimental device for kneading bread dough:



[Department of Food Science, University of Wisconsin. Photos by J-LT.]

Experiment of Boyland, Aref & Stremler

[movie 2] [movie 3]

[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]
(Simulations by M. D. Finn.)

Mathematical description

Focus on closed systems.

Periodic stirring protocols in two dimensions can be described by a homeomorphism $\varphi : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a surface.

For instance, in a closed circular container,

- φ describes the mapping of fluid elements after one full period of stirring, obtained by solving the Stokes equation;
- $\mathcal{S}$ is the disc with holes in it, corresponding to the stirring rods.

Goal: Topological characterization of φ.

Three main ingredients

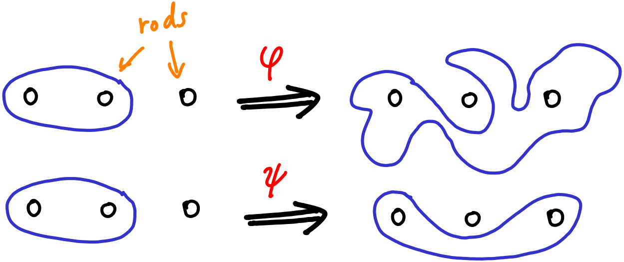
1. The Thurston–Nielsen classification theorem (idealized φ);
2. Handel's isotopy stability theorem (the real φ);
3. Topological entropy (quantitative measure of mixing).

Isotopy

φ and ψ are isotopic if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

(Defines isotopy classes.)

Convenient to think of isotopy in terms of material loops. Isotopic maps act the same way on loops (up to continuous deformation).



(Loops will always mean essential loops.)

TN classification theorem (cartoon)

φ is isotopic to a homeomorphism ψ, where ψ is in one of the following three categories:

1. finite-order (i.e., periodic);
2. reducible (can decompose into different bits);
3. pseudo-Anosov: ψ stretches all loops at an exponential rate $\log \lambda$, called the topological entropy. Any loop eventually traces out the unstable foliation.

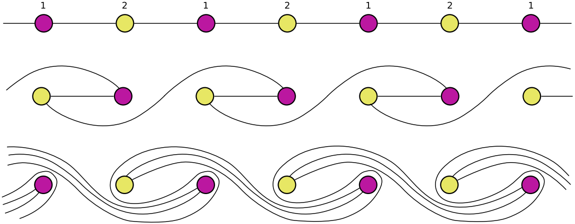
Number 3 is the one we want for good mixing

Topological ingredients

- Consider a motion of stirring elements, such as rods.
- Determine if the motion is isotopic to a pseudo-Anosov mapping.
- Compute topological quantities, such as foliation, entropy, etc.
- Analyze and optimize.

Optimization

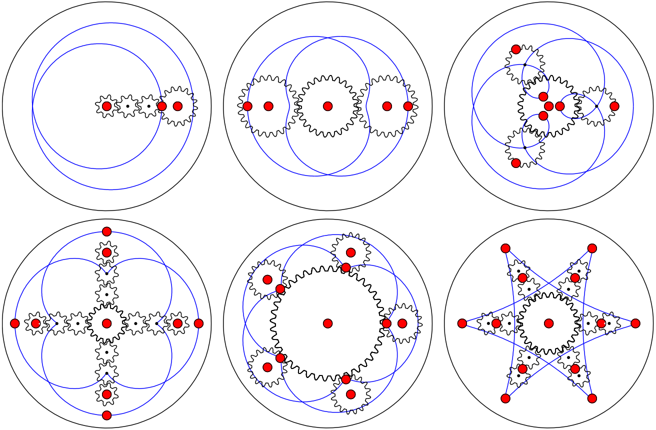
- Consider periodic lattice of rods.
- Move all the rods such that they execute the Boyland et al. (2000) rod motion (Thiffeault & Finn, 2006; Finn & Thiffeault, 2011).



- The dilatation per period is χ^2, where $\chi = 1 + \sqrt{2}$ is the Silver Ratio!
- This is optimal for a periodic lattice of two rods (Follows from D'Alessandro et al. (1999)).

Silver Mixers!

- The designs with dilatation given by the silver ratio can be realized with simple gears.
- All the rods move at once: very efficient.



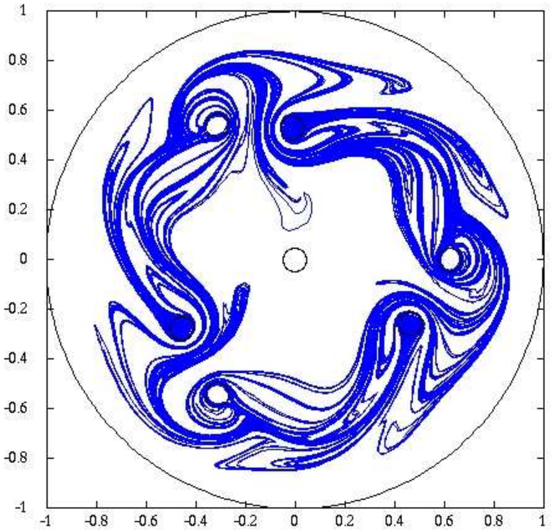
[movie 4]

Four Rods

[movie 5] [movie 6]

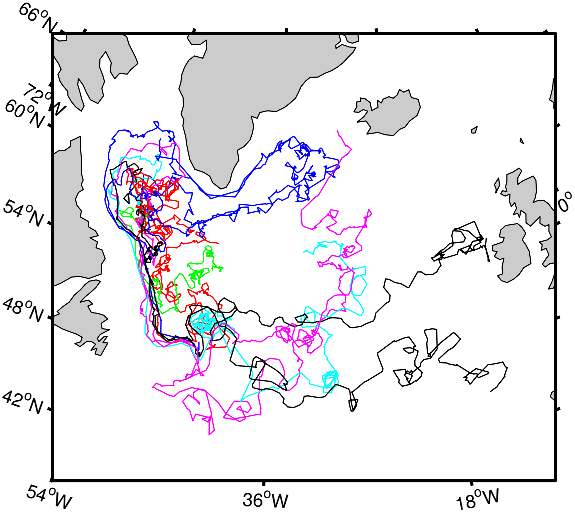
[M. D. Finn and J.-L. Thiffeault, *SIAM Review*, in press (2011)]

Six Rods



[movie 7]

Oceanic float trajectories



Oceanic floats: Data analysis

What can we measure?

- Single-particle dispersion (not a good use of all data)
- Correlation functions (what do they mean?)
- Lyapunov exponents (some luck needed!)

Another possibility:

Compute the braid group generators σ_i for the float trajectories (convert to a sequence of symbols), then look at how loops grow. Obtain a topological entropy for the motion (similar to Lyapunov exponent).

Iterating a loop

It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

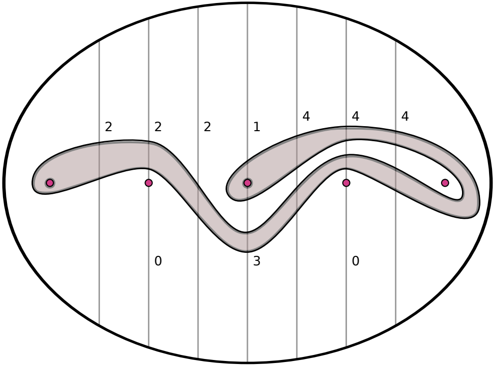
The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;
2. Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them topologically with very few numbers.

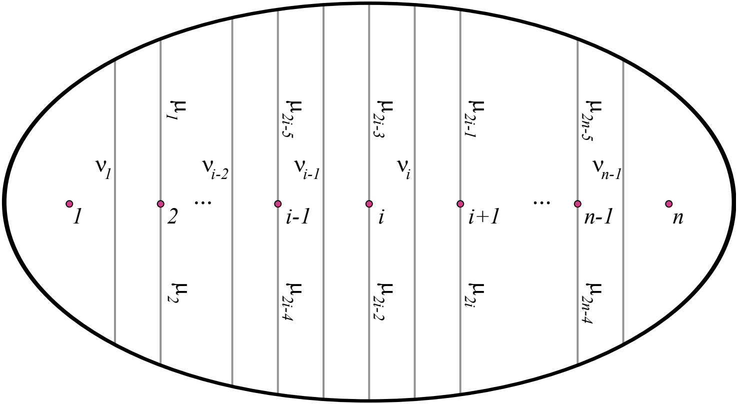
Solution to problem 1: Loop coordinates

What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the Dynnikov coordinates involve intersections with vertical lines:



Crossing numbers

Label the crossing numbers:



Dynnikov coordinates

Now take the difference of crossing numbers:

$$\begin{aligned} a_i &= \tfrac{1}{2}\left(\mu_{2i} - \mu_{2i-1}\right), \\ b_i &= \tfrac{1}{2}\left(\nu_i - \nu_{i+1}\right) \end{aligned}$$

for $i = 1, \ldots, n-2$.

The vector of length $(2n-4)$,

$$\mathbf{u} = (a_1, \ldots, a_{n-2}, b_1, \ldots, b_{n-2})$$

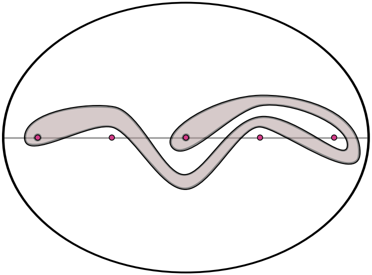
is called the Dynnikov coordinates of a loop.

There is a one-to-one correspondence between closed loops and these coordinates: you can't do it with fewer than $2n-4$ numbers.

Intersection number

A useful formula gives the minimum intersection number with the 'horizontal axis':

$$L(\mathbf{u}) = |a_1| + |a_{n-2}| + \sum_{i=1}^{n-3} |a_{i+1} - a_i| + \sum_{i=0}^{n-1} |b_i| \,,$$

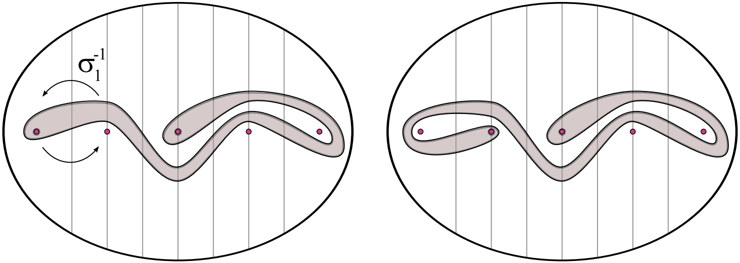


For example, the loop on the left has $L = 12$.

The crossing number grows proportionally to the the length.

Solution to problem 2: Action on coordinates

Moving the punctures according to a braid generator changes some crossing numbers:



There is an explicit formula for the change in the coordinates!

Action on loop coordinates

The update rules for σ_i acting on a loop with coordinates $(\mathbf{a},\mathbf{b})$ can be written

$$\begin{aligned} a'_{i-1} &= a_{i-1} - b^+_{i-1} - \left(b^+_i + c_{i-1}\right)^+ \,, \\ b'_{i-1} &= b_i + c^-_{i-1} \,, \\ a'_i &= a_i - b^-_i - \left(b^-_{i-1} - c_{i-1}\right)^- \,, \\ b'_i &= b_{i-1} - c^-_{i-1} \,, \end{aligned}$$

where

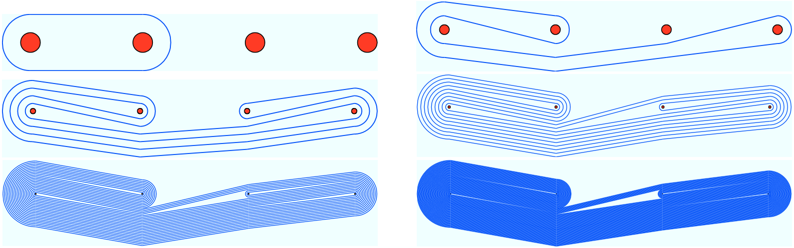
$$f^+ := \max(f,0), \qquad f^- := \min(f,0).$$

$$c_{i-1} := a_{i-1} - a_i - b^+_i + b^-_{i-1} \,.$$

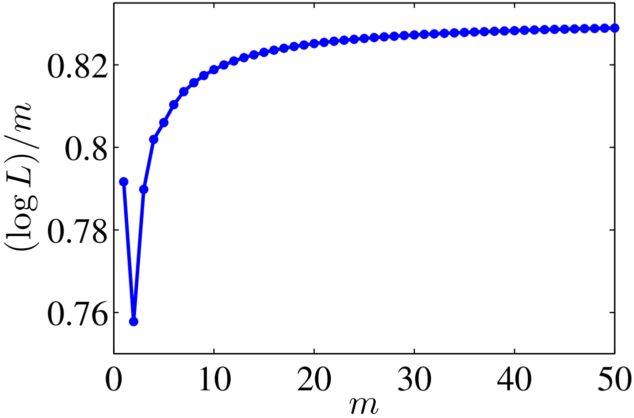
This is called a piecewise-linear action.
Easy to code up (see for example Thiffeault (2010)).

Growth of *L*

For a specific rod motion, say as given by the braid $\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1$, we can easily see the exponential growth of L and thus measure the entropy:



Growth of L (2)

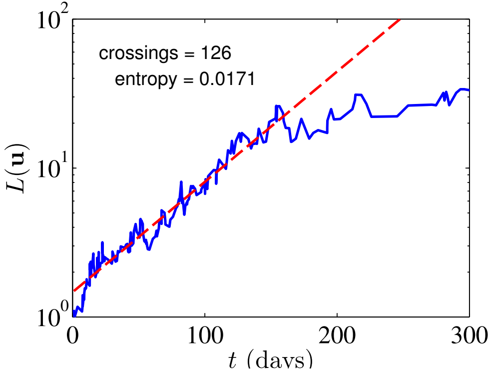


m is the number of times the braid acted on the initial loop.

[Moussafir (2006)]

Oceanic floats: Entropy

10 floats from Davis' Labrador sea data:



Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

Source: WOCE subsurface float data assembly center (2004)

Some open issues

- The nature of the isotopy between the pA and real system.
- Which orbits dominate? (They live in folds — see for instance Cerbelli & Giona (2006); Thiffeault et al. (2009))
- Sharpness of the entropy bound (progress: linked twist maps — Sturman et al. (2006)).
- Computational methods for isotopy class (random entanglements of trajectories – LCS method).
- ‘Designing’ for topological chaos.
- Combine with other measures, e.g., mix-norms (Mathew et al., 2005; Lin et al., 2010).
- 3D?! (lots of missing theory)

This work was supported by the Division of Mathematical Sciences of the US National Science Foundation, under grant DMS-0806821.

Allshouse, M. R. & Thiffeault, J.-L. 2011 Detecting Coherent Structures Using Braids. *Physica D*, in press, `http://arXiv.org/abs/1106.2231`.

Bestvina, M. & Handel, M. 1992 Train Tracks for Automorphisms of Free Groups. *Ann. Math.* **134**, 1–51.

Binder, B. J. 2010 Ghost rods adopting the role of withdrawn baffles in batch mixer designs. *Phys. Lett. A* **374**, 3483–3486.

Binder, B. J. & Cox, S. M. 2008 A Mixer Design for the Pigtail Braid. *Fluid Dyn. Res.* **40**, 34–44.

Bowen, R. 1978 Entropy and the fundamental group. In *Structure of Attractors*, volume 668 of *Lecture Notes in Math.*, pp. 21–29. New York: Springer.

Boyland, P. L., Aref, H. & Stremler, M. A. 2000 Topological fluid mechanics of stirring. *J. Fluid Mech.* **403**, 277–304.

Boyland, P. L., Stremler, M. A. & Aref, H. 2003 Topological fluid mechanics of point vortex motions. *Physica D* **175**, 69–95.

Cerbelli, S. & Giona, M. 2006 One-sided invariant manifolds, recursive folding, and curvature singularity in area-preserving nonlinear maps with nonuniform hyperbolic behavior. *Chaos Solitons Fractals* **29**, 36–47.

D'Alessandro, D., Dahleh, M. & Mezić, I. 1999 Control of mixing in fluid flow: A maximum entropy approach. *IEEE Transactions on Automatic Control* **44**, 1852–1863.

Finn, M. D. & Thiffeault, J.-L. 2011 Topological optimisation of rod-stirring devices. *SIAM Rev.*, in press, `http://arXiv.org/abs/1004.0639`.

Gouillart, E., Finn, M. D. & Thiffeault, J.-L. 2006 Topological Mixing with Ghost Rods. *Phys. Rev. E* **73**, 036311.

Handel, M. 1985 Global shadowing of pseudo-Anosov homeomorphisms. *Ergod. Th. Dynam. Sys.* **8**, 373–377.

Kobayashi, T. & Umeda, S. 2007 Realizing pseudo-Anosov egg beaters with simple mecanisms. In *Proceedings of the International Workshop on Knot Theory for Scientific Objects, Osaka, Japan*, pp. 97–109. Osaka Municipal Universities Press.

Lin, Z., Doering, C. R. & Thiffeault, J.-L. 2010 An optimal stirring strategy for passive scalar mixing. Preprint.

Mathew, G., Mezić, I. & Petzold, L. 2005 A multiscale measure for mixing. *Physica D* **211**, 23–46.

Meleshko, V. & Peters, G. W. M. 1996 Periodic points for two-dimensional Stokes flow in a rectangular cavity. *Phys. Lett. A* **216**, 87–96.

Moussafir, J.-O. 2006 On Computing the Entropy of Braids. *Func. Anal. and Other Math.* **1**, 37–46. `arXiv:math.DS/0603355`.

Stremler, M. A. & Chen, J. 2007 Generating topological chaos in lid-driven cavity flow. *Phys. Fluids* **19**, 103602.

Sturman, R., Ottino, J. M. & Wiggins, S. 2006 *The Mathematical Foundations of Mixing: The Linked Twist Map as a Paradigm in Applications: Micro to Macro, Fluids to Solids*. Cambridge, U.K.: Cambridge University Press.

Thiffeault, J.-L. 2005 Measuring Topological Chaos. *Phys. Rev. Lett.* **94**, 084502.

Thiffeault, J.-L. 2010 Braids of entangled particle trajectories. *Chaos* **20**, 017516. `arXiv:0906.3647`.

Thiffeault, J.-L. & Finn, M. D. 2006 Topology, Braids, and Mixing in Fluids. *Phil. Trans. R. Soc. Lond. A* **364**, 3251–3266.

Thiffeault, J.-L., Finn, M. D., Gouillart, E. & Hall, T. 2008 Topology of Chaotic Mixing Patterns. *Chaos* **18**, 033123. `arXiv:0804.2520`.

Thiffeault, J.-L., Gouillart, E. & Finn, M. D. 2009 The Size of Ghost Rods. In L. Cortelezzi & I. Mezić, editors, *Analysis and Control of Mixing with Applications to Micro and Macro Flow Processes*, volume 510 of *CISM International Centre for Mechanical Sciences*, pp. 339–350. Vienna: Springer. `arXiv:nlin/0510076`.

Thurston, W. P. 1988 On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Am. Math. Soc.* **19**, 417–431.

Vikhansky, A. 2003 Chaotic advection of finite-size bodies in a cavity flow. *Phys. Fluids* **15**, 1830–1836.