AOS Informal Lecture: 2011/04/07

Mixing in the Presence of Sources and Sinks

Stirring: mechanical action (cause)
Mixing: homogenization of a scalar (effect)

$\theta(\underline{x}, t)$ = concentration, $\underline{u}(\underline{x}, t)$ given

$$\partial_t \theta + \underline{u} \cdot \nabla\theta = \kappa \nabla^2 \theta, \quad \nabla \cdot \underline{u} = 0 \qquad \text{(AD)}$$

in Ω a
- bounded domain with zero-flux conditions
- OR periodic domain

$$\left.\begin{aligned} \hat{n} \cdot \nabla\underline{\theta} &= 0 \\ \hat{n} \cdot \underline{u} &= 0 \end{aligned}\right\} \text{ on boundary } \partial\Omega$$

Multiply AD by $m\,\theta^{m-1}$, integrate:

$$\langle m\theta^{m-1} \partial_t \theta \rangle = \partial_t \langle \theta^m \rangle$$

$$\langle m\theta^{m-1} \underline{u} \cdot \nabla\theta \rangle = \langle \underline{u} \cdot \nabla\theta^m \rangle \overset{\nabla \cdot \underline{u} = 0}{=} \langle \nabla \cdot (\underline{u}\theta^m) \rangle$$

$$= \int_{\partial\Omega} \theta^m \underbrace{\underline{u} \cdot \hat{n}}_{0!} \, dS = 0$$

$$\langle m\theta^{m-1} \kappa \nabla^2 \theta \rangle = \kappa m \langle \nabla \cdot (\theta^{m-1} \nabla\theta) - \nabla\theta^{m-1} \cdot \nabla\theta \rangle$$

$$= \kappa m \int_{\partial\Omega} \theta^{m-1} \underbrace{\nabla\theta \cdot \hat{n}}_{0!} \, dS - \kappa\, m(m-1) \langle \theta^{m-2} |\nabla\theta|^2 \rangle$$

$$\boxed{\partial_t \langle \theta^m \rangle = -\kappa\, m(m-1) \langle \theta^{m-2} |\nabla\theta|^2 \rangle}$$

$m=0$ is trivial

$m=1$: $\partial_t \langle \theta \rangle = 0$ Total amount of θ is conserved

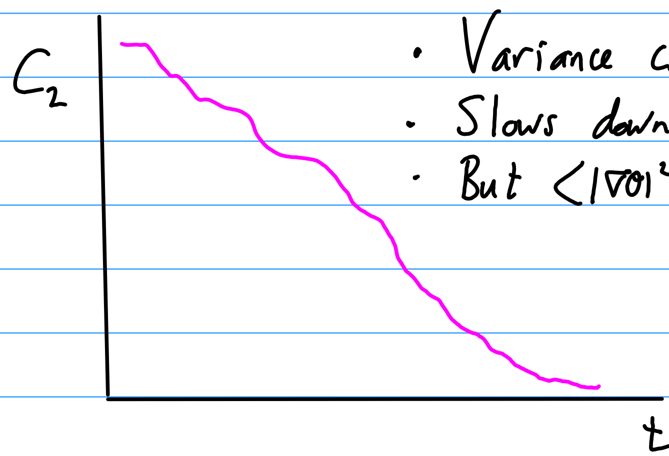
$m=2$: $\partial_t \langle \theta^2 \rangle = -2\kappa \langle |\nabla\theta|^2 \rangle$ $\langle \theta^2 \rangle$ non-increasing!

Let variance $\mathrm{Var} = C_2 = \langle \theta^2 \rangle - \langle \theta \rangle^2 / \text{Volume}$

constant

$$\partial_t C_2 = -2\kappa \langle |\nabla\theta|^2 \rangle$$

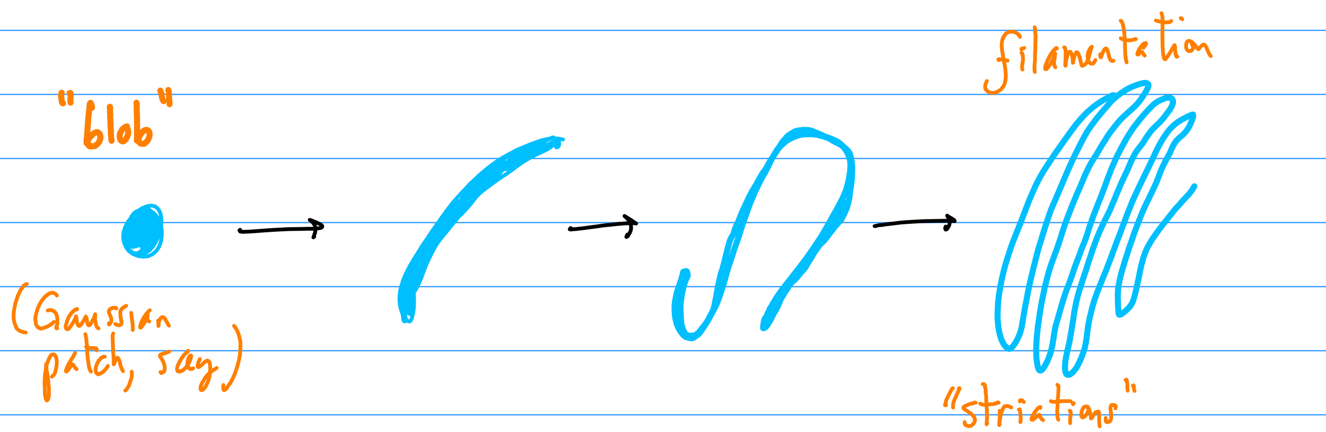
Scenario:



- Variance can only decrease.
- Slows down as $\langle |\nabla\theta|^2 \rangle \to 0$
- But $\langle |\nabla\theta|^2 \rangle = 0$ iff $\theta = $ constant.

in some sense

But of course the variance equation is not closed: it depends on $\nabla\theta$.
What happens when you stir?



This hints at the answer: stirring increases $\nabla\theta$

$$\partial_t \langle \theta^2 \rangle = -2\kappa \underbrace{\langle |\nabla\theta|^2 \rangle}_{\text{this becomes larger as we stir}}$$

So variance is a good measure of mixing.

$\theta_0 \equiv 0$!

Another: mix-norm $\langle |\nabla^{-1}\theta|^2 \rangle$ $\quad (\nabla^{-1}\theta)_k = \dfrac{-ik}{|k|^2}\theta_k$

smoothing

$\nabla \cdot \nabla^{-1}\theta = \theta$

gradient norm $\langle |\nabla\theta|^2 \rangle$

Sources and Sinks:

$$\partial_t \theta + \underline{u} \cdot \nabla \theta = \kappa \nabla^2 \theta + \underbrace{s(\underline{x}, t)}_{\substack{\text{sources/sinks} \\ >0 \quad <0}} \qquad \text{(ADs)}$$

$$(\nabla \cdot \underline{u} = 0)$$

Assume: $\int_\Omega s(\underline{x}, t)\, d\Omega = 0$ (otherwise subtract the mean)

More convenient to think of hot/cold
sources sinks

For simplicity, restrict to time-independent $s(\underline{x})$.
Then system achieves a steady-state: (unlike decaying problem)

$$\underline{u} \cdot \nabla \theta = \kappa \nabla^2 \theta + s \qquad \text{Let } \mathcal{L} \equiv \underline{u} \cdot \nabla - \kappa \nabla^2$$

$$\hookrightarrow \quad \mathcal{L}\theta = s \quad \text{or} \quad s = \underbrace{\mathcal{L}^{-1}}_{\text{integral operator}} \theta$$

Note that $\kappa \neq 0$ is needed to reach steady-state.

So, assuming the system has reached a steady-state, how do we measure the "quality of mixing"?

Can look at norm $\langle \theta^2 \rangle$

But what do we compare to?

One possibility: $\dfrac{\langle \theta^2 \rangle}{\langle s^2 \rangle}$ Pretty good, but has units of inverse time.

Prefer mixing enhancement factor: (or mixing efficiency)

$$\boxed{\mathcal{E}_0^2 = \frac{\langle\tilde{\theta}^2\rangle}{\langle\theta^2\rangle}}$$

$$\tilde{\mathcal{L}} = -\kappa\nabla^2$$

$$\tilde{\mathcal{L}}\tilde{\theta} = s$$

$\tilde{\theta}$ is the solution in the absence of stirring. (purely diffusive)

Since $\langle\theta^2\rangle$ is usually decreased by stirring, $\mathcal{E}_0$ measures the enhancement over the pure-diffusion state.

Several properties given in Doering & T, Shaw, Doering, & T.

For instance, can we have $\mathcal{E}_0 < 1$, i.e., can stirring ever be worse than not stirring?

Consider also: $\mathcal{E}_1 = \dfrac{\langle|\nabla\tilde{\theta}|^2\rangle}{\langle|\nabla\theta|^2\rangle}$, $\mathcal{E}_{-1} = \dfrac{\langle|\nabla^{-1}\tilde{\theta}|^2\rangle}{\langle|\nabla^{-1}\theta|^2\rangle}$.

$$\tilde{\theta} = \tilde{\mathcal{L}}^{-1}s = (-\kappa\nabla^2)^{-1}s = -\kappa^{-1}\nabla^{-2}s \Rightarrow \nabla\tilde{\theta} = -\kappa^{-1}\nabla^{-1}s.$$

Also: $\mathcal{L}\theta = s \Rightarrow \langle\theta\,\mathcal{L}\theta\rangle = \langle s\theta\rangle \qquad \langle\cdot\rangle = \int_\Omega \cdot\, d\Omega$

$$\langle\theta\,\cancel{u\cdot\nabla\theta}\rangle - \kappa\langle\theta\nabla^2\theta\rangle = \langle s\theta\rangle$$

$= \langle\nabla\cdot(u\theta^2/2)\rangle = 0$

$$\kappa\langle|\nabla\theta|^2\rangle = \langle\theta s\rangle = \langle\theta\,\nabla\cdot\nabla^{-1}s\rangle$$

$$= -\langle\nabla\theta\cdot\nabla^{-1}s\rangle = \kappa\langle\nabla\theta\cdot\nabla\tilde{\theta}\rangle$$

$$\langle|\nabla\theta|^2\rangle = \underbrace{\langle\nabla\theta\cdot\nabla\tilde{\theta}\rangle}_{=1} \le \langle|\nabla\theta|^2\rangle^{1/2}\langle|\nabla\tilde{\theta}|^2\rangle^{1/2}$$

Cauchy-Schwartz inequality

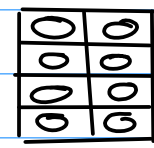
$$\therefore \langle |\nabla\theta|^2\rangle \leq \langle |\nabla\tilde{\theta}|^2\rangle \iff \boxed{\mathcal{E}_1 \geq 1}$$

This is somewhat counter-intuitive: gradients are usually increased by stirring! However, here we're talking about gradients in a steady-state, affected by diffusion.

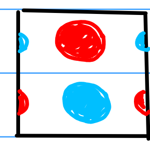
What about the other ones, $\mathcal{E}_q$, $q \neq 1$? Do we have $\mathcal{E}_q \geq 1$?

We tried and failed to prove this, because it isn't true. Following a challenge by Charlie Doering, Jeff Weiss came up with something like:

$$\underline{u} = (2\sin x \cos 2y,\ -\cos x \sin 2y)$$



$$s = (\cos x - \tfrac{1}{2})\sin y$$



(Péclet $=4$)

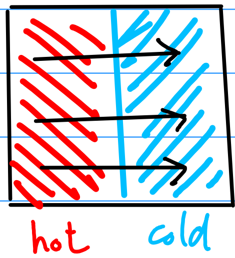
This manages to "concentrate" the source sink distribution more that under pure diffusion, and

$$\mathcal{E}_0 \simeq .978, \qquad \mathcal{E}_{-1} \simeq .945$$

Slightly less than 1! Not a dramatic effect, but it's there!

OPTIMIZATION: What kinds of flow give the largest $\mathcal{E}_0$, given source/sink distribution $s(\underline{x})$? (FIXED ENERGY)

Surprising example: $s(\underline{x}) = \sin x$ (periodic B.C.)



optimal: $\underline{u} = U\hat{x}$ Constant flow!

(see Shaw-T-Doering, Plasting-Young)

This example demonstrates that with body sources the best stirring has more to do with transport than with creation of small scales.

SOLVE NUMERICALLY for more complicated sources.