Velocity fluctuations in suspensions of swimming microorganisms

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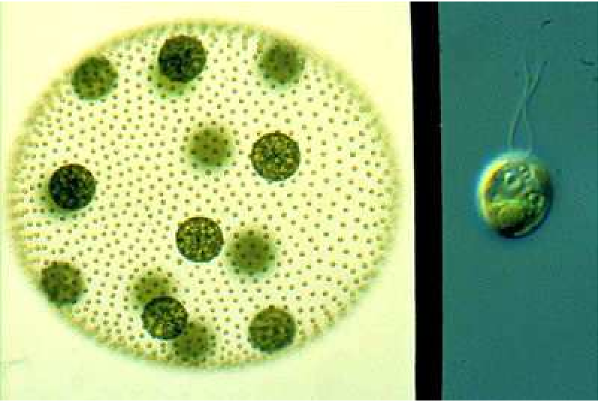
Microswimmers

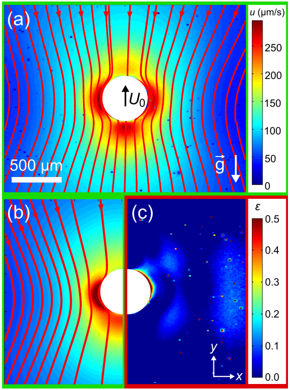


Volvox carteri and *Chlamydomonas reinhardtii*

Chlammy is single-celled ($\sim 5\ \mu$m); *Volvox* consists of several thousand chlammy-like cells ($\sim 200\ \mu$m).

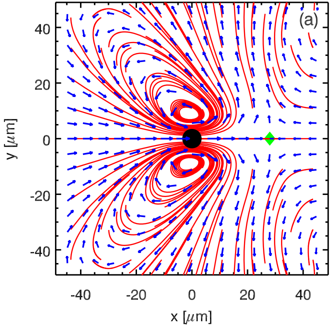
Velocity field

Volvox



Drescher *et al.* (2010)

Chlammy

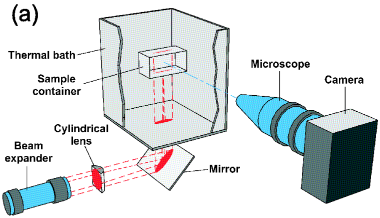


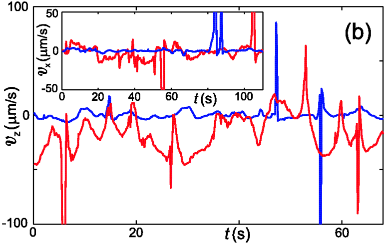
Guasto *et al.* (2010)

$$\mathbf{v}_{\mathsf{fit}}(\boldsymbol{r}) = \frac{A_{\mathsf{St}}}{r}\left(\mathbf{I}+\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right)\cdot\hat{\boldsymbol{z}} + \frac{A_{\mathsf{str}}}{r^2}\left(1-3(z/r)^2\right)\hat{\boldsymbol{r}} + \frac{A_{\mathsf{sd}}}{3r^3}\left(\mathbf{I}-3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right)\cdot\hat{\boldsymbol{z}}$$

Measuring velocity fluctuations

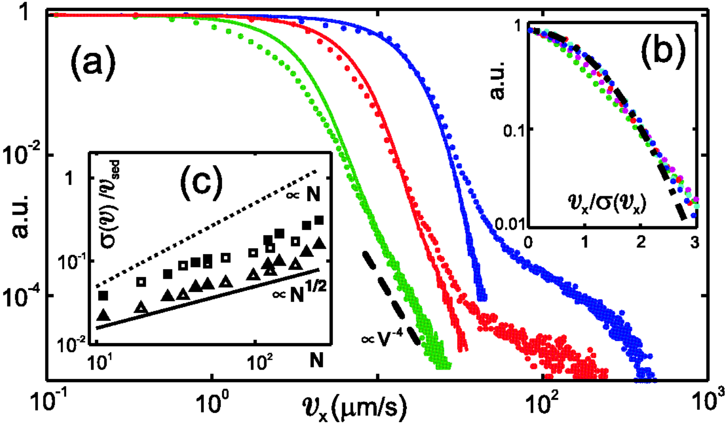
Experiments of Rushkin *et al.* (2010) using *Volvox*:





Velocity measured at a fixed point.

PDF of velocity fluctuations



Number of *Volvox*: green (11), red (42), blue (210)

Circles are experiments: note cut-off at large v_x.

Drescher, K. D., Goldstein, R. E., Michel, N., Polin, M. & Tuval, I. 2010 Direct measurement of the flow field around swimming microorganisms. *Phys. Rev. Lett.* **105**, 168101.

Guasto, J. S., Johnson, K. A. & Gollub, J. P. 2010 Oscillatory flows induced by microorganisms swimming in two-dimensions. *Phys. Rev. Lett.* **105**, 168102.

Hernandez-Ortiz, J. P., Dtolz, C. G. & Graham, M. D. 2006 Transport and collective dynamics in suspensions of confined swimming particles. *Phys. Rev. Lett.* **95**, 204501.

Lin, Z., Thiffeault, J.-L. & Childress, S. 2010 Stirring by squirmers. *J. Fluid Mech.* In press.

Rushkin, I., Kantsler, V. & Goldstein, R. E. 2010 Fluid velocity fluctuations in a suspension of swimming protists. *Phys. Rev. Lett.* In press.

Thiffeault, J.-L. & Childress, S. 2010 Stirring by swimming bodies. *Phys. Lett. A* **374**, 3487–3490.

Underhill, P. T., Hernandez-Ortiz, J. P. & Graham, M. D. 2008 Diffusion and spatial correlations in suspensions of swimming particles. *Phys. Rev. Lett.* **100**, 248101.

Weiss, J. B., Provenzale, A. & McWilliams, J. C. 1998 Lagrangian dynamics in high-dimensional point-vortex systems. *Phys. Fluids* **10** (8), 1929–1941.

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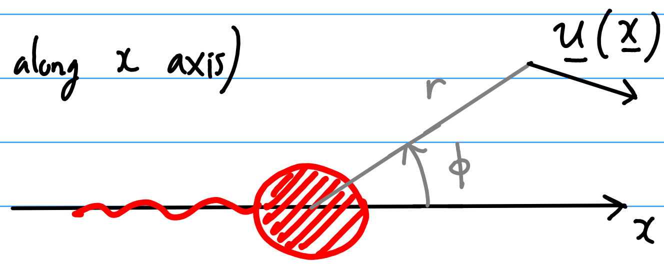
Velocity field due to one swimmer:

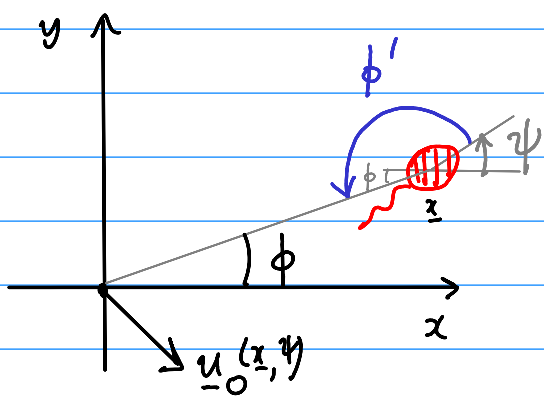
(oriented along x axis)



$$\underline{x} = r(\cos\phi, \sin\phi)$$

$$\underline{u} = u(r,\phi)\hat{x} + v(r,\phi)\hat{y}$$

Now assume this swimmer is at position $\underline{x}$, angle ψ: what is the velocity field induced by the swimmer at O, the origin?



$$\phi' = \phi - \psi - \pi$$

Rotate ϕ by $-(\psi+\pi)$.

After also rotating direction, get

$$\underline{u}_0(\underline{x},\psi) = R_\psi \, \underline{u}\left(R^{-1}_{\psi+\pi}\,\underline{x}\right)$$

where $R_\psi = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$ is a rotation matrix.

Now, the velocity at the origin is due to a superposition of N swimmers.

$$u_N = \sum_{k=1}^{N} u_0(\underline{x}_k, \psi_k)$$

Consider only horizontal component

What is the probability that $u_N = u$ for N swimmers?

$$P_N(u) = \int \delta\left(u_0(\underline{x}_1, \psi_1) + \cdots + u_0(\underline{x}_N, \psi_N) - u\right)$$
$$\times \quad \frac{d\psi_1}{2\pi} \cdots \frac{d\psi_N}{2\pi} \; \frac{dV_1}{V} \cdots \frac{dV_N}{V}$$

The characteristic function is the Fourier transform

$$e^{-S_N(k)} = \int P_N(u)\, e^{-iku}\, du$$

Make use of δ-function:

$$e^{-S_N(k)} = \int e^{-ik\left(u_0(\underline{x}_1,\psi_1) + \cdots + u_0(\underline{x}_N,\psi_N)\right)} \frac{d\psi_1}{2\pi} \cdots \frac{d\psi_N}{2\pi} \frac{dV_1}{V} \cdots \frac{dV_N}{V}$$

$$= \left(\frac{1}{2\pi V} \int e^{-iku_0(\underline{x},\psi)}\, d\psi\, dV\right)^N = e^{-Ns(k)}$$

$$u_0(\underline{x},\psi) = \cos\psi\, u(r, \phi - \psi - \pi) - \sin\psi\, v(r, \phi - \psi - \pi)$$

$$dV = r\, dr\, d\phi$$

Let $\phi' = \phi - \psi - \pi$.

$$e^{-s(k)} = \frac{1}{2\pi V} \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi' \int r\,dr \; e^{-ik\left(\cos\psi\, u(r,\phi') - \sin\psi\, v(r,\phi')\right)}$$

Now let $u(r,\phi) = U(r,\phi)\cos\alpha(r,\phi)$
$v(r,\phi) = U(r,\phi)\sin\alpha(r,\phi)$

$$e^{-s(k)} = \frac{1}{2\pi V}\int_0^{2\pi} d\phi' \int r\,dr \int_0^{2\pi} d\psi\, e^{-ik\,U(r,\phi')\cos(\psi+\alpha(r,\phi'))}$$

Now let $\psi' = \psi + \alpha(r,\phi')$.

$$e^{-s(k)} = \frac{1}{2\pi V}\int_0^{2\pi} d\phi' \int r\,dr \int_0^{2\pi} d\psi'\, e^{-ik\,U(r,\phi')\cos\psi'}$$

Bessel func.

$$= \frac{1}{V}\int_0^{2\pi} d\phi' \int r\,dr\; J_0\left(k\,U(r,\phi')\right)$$

$$\boxed{e^{-s(k)} = \frac{1}{V}\int dV\; J_0\left(k\,U(\underline{x})\right)}$$

Only magnitude matters, since directions uniformly distributed

Actually more convenient to rewrite as:

$$e^{-s(k)} = 1 - \frac{1}{V}\int dV\left(1 - J_0\left(k\,U(\underline{x})\right)\right)$$

Now consider a circular domain, and assume

$$U(\underline{x}) = \frac{A}{r^n}$$

$n = 1$: point vortices or sources
$n = 2$: point dipole

$$e^{-s(k)} = 1 - \frac{2\pi}{V}\int_0^R \left(1 - J_0(kA/r^n)\right) r\, dr$$

For small r, $J_0(\infty) \to 0$, so integral converges

For large r, $1 - J_0(kA/r^n) = \frac{k^2A^2}{4}\frac{1}{r^{2n}} + O\left(\left(\frac{kA}{r^n}\right)^4\right)$,

so integrand $\sim r^{1-2n}$.

The limit $R \to \infty$ thus diverges for $1 - 2n \geq -1 \Leftrightarrow n \leq 1$.

CANNOT ASSUME INFINITE VOLUME FOR VORTICES! $(n=1)$

But in a way this makes thing easier: for large R, the value of the integral is dominated by large R:

$$e^{-s(k)} \approx 1 - \frac{2\pi}{V}\frac{k^2A^2}{4}\int_\ell^R r^{1-2n}\, dr, \quad n \leq 1$$

cut-off: come back later. (needed for $n=1$)

$$= 1 - \frac{\pi k^2A^2}{2V}\log(R/\ell)$$

Recall $e^{-S_N(k)} = e^{-Ns(k)}$, so

number density of vortices

$$N = cV$$

$$e^{-S_N(k)} = \left(1 - \frac{\pi k^2A^2}{2V}\log(R/\ell)\right)^N$$

Gaussian!

$$= \left(1 - \frac{\pi k^2A^2c}{2N}\log(R/\ell)\right)^N \approx \exp\left(-\frac{\pi k^2A^2c}{2}\log\left(\frac{R}{\ell}\right)\right)$$

for large N.

Now go back to u: $P_N(u) = \frac{1}{2\pi}\int e^{-S_N(k)} e^{iku}\, dk$

Get a Gaussian PDF with mean zero and variance

$$\boxed{\sigma^2 = \pi A^2 c \log(R/\ell)}$$

$\sim N \log N$, as in Weiss, Provenzale, McWilliams 198.

For $n > 1$, can take limit $R \to \infty$: (implies $N \to \infty$, constant c)

$$e^{-s(k)} = 1 - \frac{2\pi}{V}\int_0^\infty \left(1 - J_0(kA/r^n)\right) r\, dr \quad , \quad n > 1$$

$$= 1 + \frac{2\pi}{V}\frac{1}{2}\left(\frac{|k|A}{2}\right)^{2/n} \frac{\Gamma(-1/n)}{\Gamma(1/n)}$$

$$e^{-S_N(k)} = e^{-Ns(k)} \approx \exp\left(-\pi c\left(\frac{|k|A}{2}\right)^{2/n}\frac{\Gamma(-1/n)}{\Gamma(1/n)}\right)$$

for large N.

$$= \exp\left(-C_n |k|^{2/n}\right) \quad , \quad n > 1$$

$n = 2$ is particularly simple: $e^{-S_N(k)} = e^{-\pi c A|k|}$

(inviscid cylinders)

Inverse transform:

$$\boxed{P(u) = \frac{Ac}{(\pi A c)^2 + u^2}}$$

Lorentzian

Note that for $n > 1$ the second moment diverges, as observed by Rushkin (in 3D). Cure this with a finite-size cut-off:

$$\sigma^2 = \langle u^2 \rangle = \int_0^{u_{max}} P(u)\, u^2\, du$$

with $u_{max} = \dfrac{A}{\ell^n}$, where ℓ is the organism size.

For $n=2$, get $\boxed{\sigma^2 = \left(\frac{A}{\ell}\right)^2 c - \pi A^2 c^2 \operatorname{arccot}(\pi c \ell^2)}$

Since $c\ell^2 \ll 1$ (low density), $\sigma^2 \approx (A/\ell)^2 c$.

Note that we can call $u_{max} = \dfrac{A}{\ell^2} = U$, the swimming velocity.
Then:

$$\boxed{\sigma^2 \simeq U^2 \ell^2 c}$$

Strong dependence on cylinder radius

$\phi = \pi \ell^2 c$ = volume fraction

Can we use σ^2 to estimate effective diffusivity?

vortices: yes but no (see Weiss et al.)

dipoles: definitely no

Stokeslet: maybe...

Green-Kubo

$$D = \lim_{T\to\infty} \int_0^T \langle u(0)\, u(t) \rangle\, dt$$

$\approx \langle u^2(0) \rangle T$ if δ-correlated

3D: boundary between divergence and convergence at large r is $n = 3/2$ (fractional)

$n = 1$: Stokeslet (Gaussian) ← strong volume dependence

$n = 2$. stresslet

$n = 3$: source doublet

Other interesting extensions:

- "mixed" singularities (Volvox)
- noise
- images
- Oseen corrections
- time-dependence
- match to data and simulations