Topological methods for stirring and mixing

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The Taffy Puller

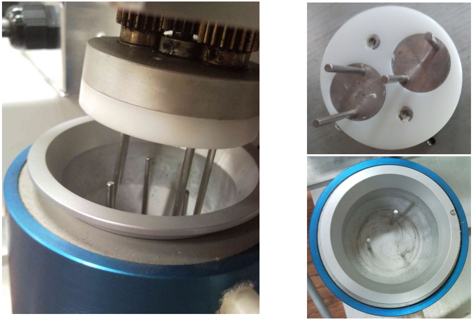
This may not look like it has much to do with stirring, but notice how the taffy is stretched and folded exponentially.

Often the hydrodynamics are less important than the precise nature of the rod motion.

[movie 1]

The mixograph

Experimental device for kneading bread dough:



[Department of Food Science, University of Wisconsin. Photos by J-LT.]

Experiment of Boyland, Aref & Stremler

[movie 2] [movie 3]

[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]
(Simulations by M. D. Finn.)

Mathematical description

Focus on closed systems.

Periodic stirring protocols in two dimensions can be described by a homeomorphism $\varphi : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a surface.

For instance, in a closed circular container,

- φ describes the mapping of fluid elements after one full period of stirring, obtained by solving the Stokes equation;
- $\mathcal{S}$ is the disc with holes in it, corresponding to the stirring rods.

Goal: Topological characterization of φ.

Three main ingredients

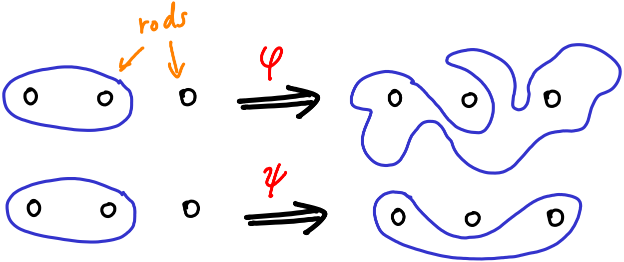
1. The Thurston–Nielsen classification theorem (idealized φ);
2. Handel's isotopy stability theorem (the real φ);
3. Topological entropy (quantitative measure of mixing).

Isotopy

φ and ψ are isotopic if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

(Defines isotopy classes.)

Convenient to think of isotopy in terms of material loops. Isotopic maps act the same way on loops (up to continuous deformation).



(Loops will always mean essential loops.)

Thurston–Nielsen classification theorem

φ is isotopic to a homeomorphism ψ, where ψ is in one of the following three categories:

1. finite-order: for some integer $k > 0$, $\psi^k \simeq$ identity;
2. reducible: ψ leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;
3. pseudo-Anosov: ψ leaves invariant a pair of transverse measured singular foliations, $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$, such that $\psi(\mathcal{F}^{\mathrm{u}},\mu^{\mathrm{u}}) = (\mathcal{F}^{\mathrm{u}},\lambda\,\mu^{\mathrm{u}})$ and $\psi(\mathcal{F}^{\mathrm{s}},\mu^{\mathrm{s}}) = (\mathcal{F}^{\mathrm{s}},\lambda^{-1}\mu^{\mathrm{s}})$, for dilatation $\lambda \in \mathbb{R}_+$, $\lambda > 1$.

The three categories characterize the isotopy class of φ.

TN classification theorem (cartoon)

φ is isotopic to a homeomorphism ψ, where ψ is in one of the following three categories:

1. finite-order (i.e., periodic);
2. reducible (can decompose into different bits);
3. pseudo-Anosov: ψ stretches all loops at an exponential rate $\log\lambda$, called the topological entropy. Any loop eventually traces out the unstable foliation.

Number 3 is the one we want for good mixing

Handel's isotopy stability theorem

The TN classification tells us about a simpler map ψ, the TN representative. What about the original map φ?

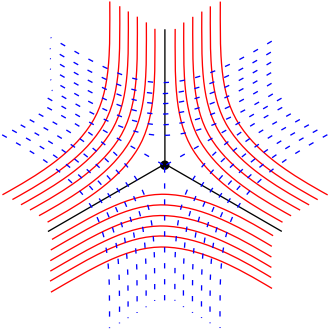
Theorem (Handel, 1985): If ψ is pseudo-Anosov and isotopic to $\varphi : \mathcal{S} \to \mathcal{S}$, then there is a compact, φ-invariant set, $\mathcal{Y} \subset \mathcal{S}$, and a continuous, onto mapping $\alpha : \mathcal{Y} \to \mathcal{S}$, so that $\alpha\varphi = \psi\alpha$.

This is called a semiconjugacy (α not generally invertible).

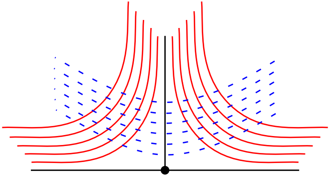
Succinctly: the dynamics of the pseudo-Anosov map 'survive' isotopy, and so φ is at least as complicated as ψ. (In particular, it has at least as much topological entropy.)

A singular foliation

The ‘pseudo’ in pseudo-Anosov refers to the fact that the foliations can have a finite number of pronged singularities.

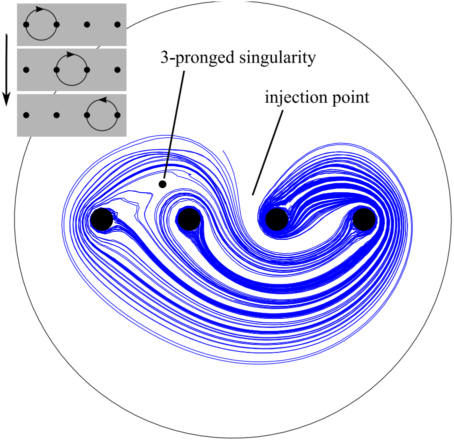


3-pronged singularity



Boundary singularity

Visualizing a singular foliation



(Thiffeault et al., 2008)

- A four-rod stirring protocol;
- Material lines trace out leaves of the unstable foliation;
- Each rod has a 1-pronged singularity.
- One 3-pronged singularity in the bulk.
- One injection point (top): corresponds to boundary singularity;

Topological ingredients

- Consider a motion of stirring elements, such as rods.
- Determine if the motion is isotopic to a pseudo-Anosov mapping.
- Compute topological quantities, such as foliation, entropy, etc.
- Analyze and optimize.

Ghost rods: Periodic orbits that stir

When trying to explain the stretching observed in a simulation, physical rods are usually not enough:

'ghost rods'

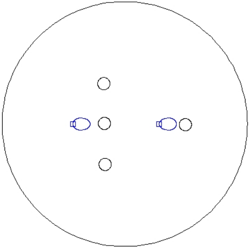
solid rods

(Gouillart et al., 2006; Stremler & Chen, 2007; Binder & Cox, 2008; Thiffeault et al., 2008; Binder, 2010; Thiffeault, 2010)

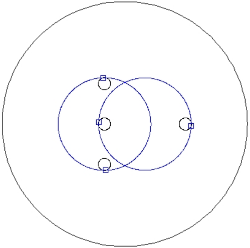
Related: Boyland et al. (2003); Vikhansky (2003); Thiffeault (2005)

So where are the ghost rods?

material line



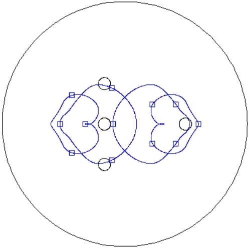
2 period-1 orbits



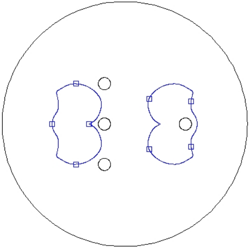
2 period-2 orbits

(Joint work with Sarah Tumasz)

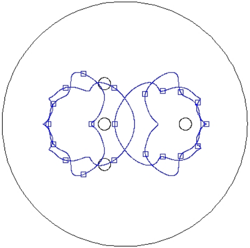
From period 3 to 8



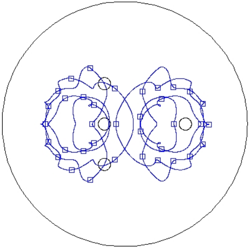
4 period-3 orbits



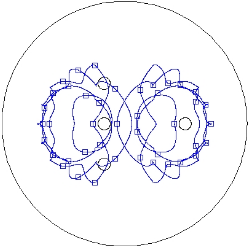
2 period-4 orbits



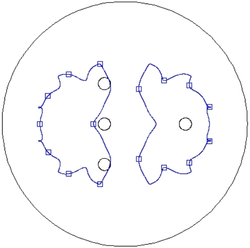
4 period-5 orbits



6 period-6 orbits



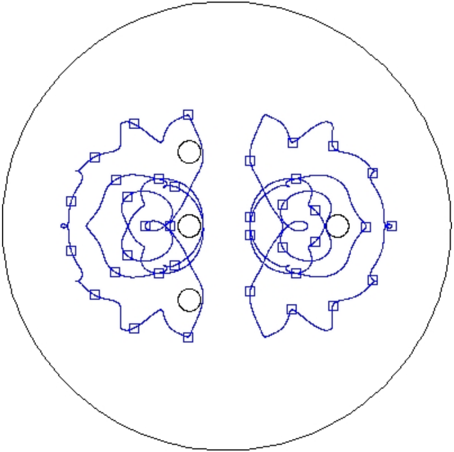
6 period-7 orbits



2 period-8 orbits

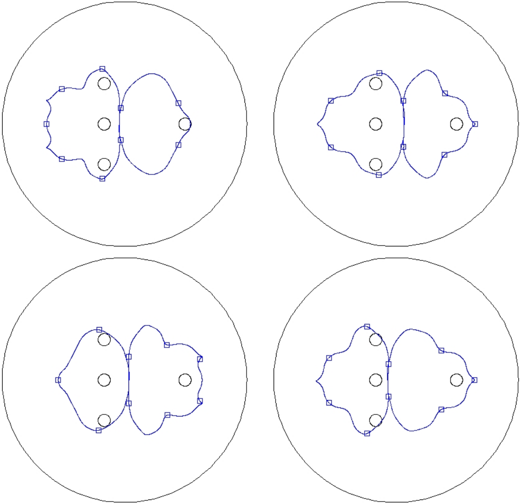
Period 9

8 period-9 orbits: 4 of the same type as before. . .



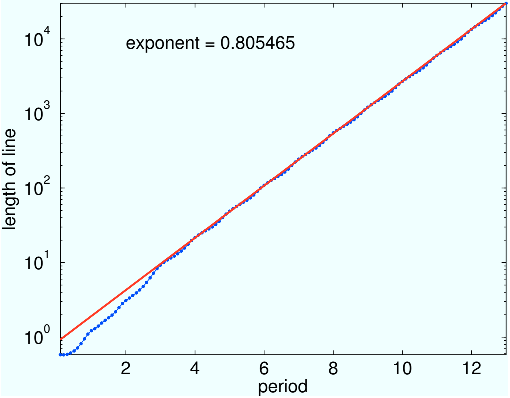
Period 9: Figure-eight orbits!

. . . and 4 new ones



[movie 4]

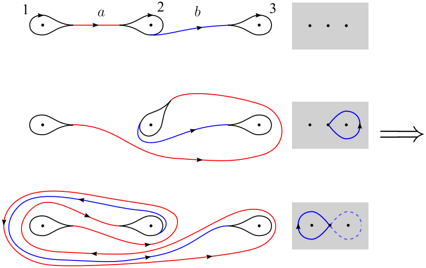
Growth rate of material lines



- The blue curve is the length of a material line;
- The red curve is the entropy of the pA of the four period-9 figure-eight orbits.

The pA entropy is the minimum stretching rate imparted on material lines if the periodic orbits were ‘rods’.

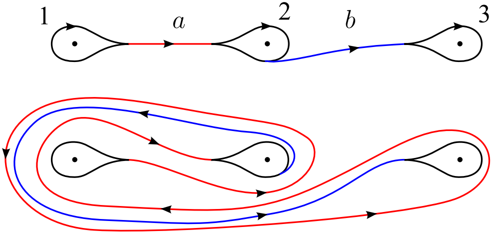
Train tracks



exp. by E. Gouillart and O. Dauchot

Thurston introduced train tracks as a way of characterizing the measured foliation. The name stems from the 'cusps' that look like train switches.

Train track map for figure-eight



$$a \mapsto a\,\bar{2}\,\bar{a}\,\bar{1}\,a\,b\,\bar{3}\,\bar{b}\,\bar{a}\,1\,a\,, \qquad b \mapsto \bar{2}\,\bar{a}\,\bar{1}\,a\,b$$

Easy to show that this map is efficient: under repeated iteration, cancellations of the type $a\,\bar{a}$ or $b\,\bar{b}$ never occur.

There are algorithms, such as Bestvina & Handel (1992), to find efficient train tracks. (Toby Hall has an implementation in C++.)

Topological Entropy

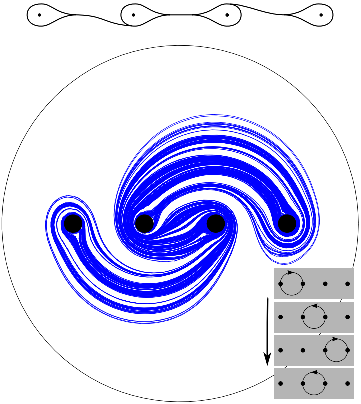
As the TT map is iterated, the number of symbols grows exponentially, at a rate given by the topological entropy, $\log \lambda$. This is a lower bound on the minimal length of a material line caught on the rods.

Find from the TT map by Abelianizing: count the number of occurences of a and b, and write as matrix:

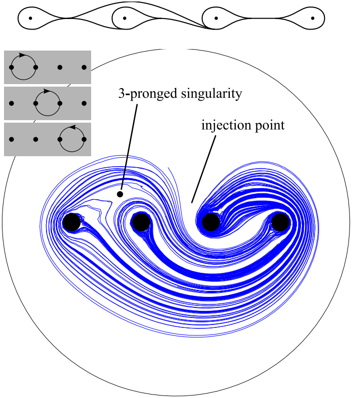
$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The largest eigenvalue of the matrix is $\lambda = (1 + \sqrt{2})^2 \simeq 5.83$. Hence, asymptotically, the length of the 'blob' is multiplied by 5.83 for each full stirring period.

Two types of stirring protocols for 4 rods



2 injection points

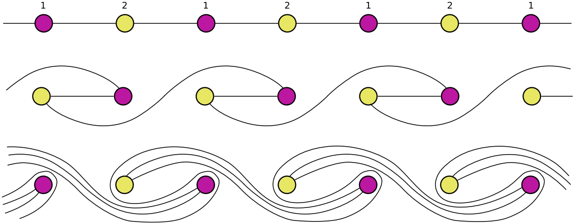


1 injection pt, 1 3-prong sing.

Topological index formulas allow us to classify train tracks, and thus stirring protocols. (Thiffeault et al., 2008)

Optimization

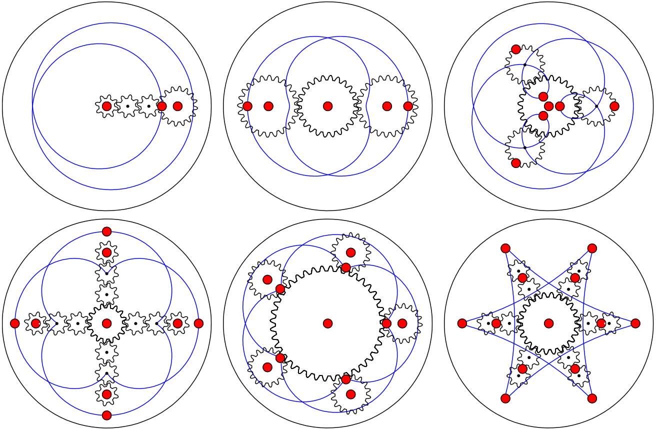
- Consider periodic lattice of rods.
- Move all the rods such that they execute the Boyland et al. (2000) rod motion (Thiffeault & Finn, 2006; Finn & Thiffeault, 2011).



- The dilatation per period is χ^2, where $\chi = 1 + \sqrt{2}$ is the Silver Ratio!
- This is optimal for a periodic lattice of two rods (Follows from D'Alessandro et al. (1999)).

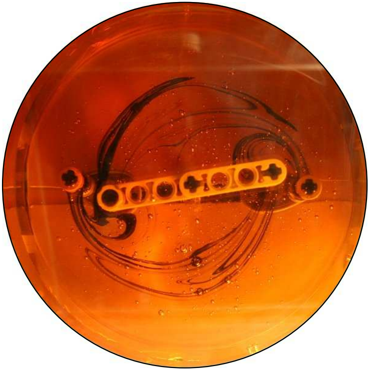
Silver Mixers!

- The designs with dilatation given by the silver ratio can be realized with simple gears.
- All the rods move at once: very efficient.



[movie 5]

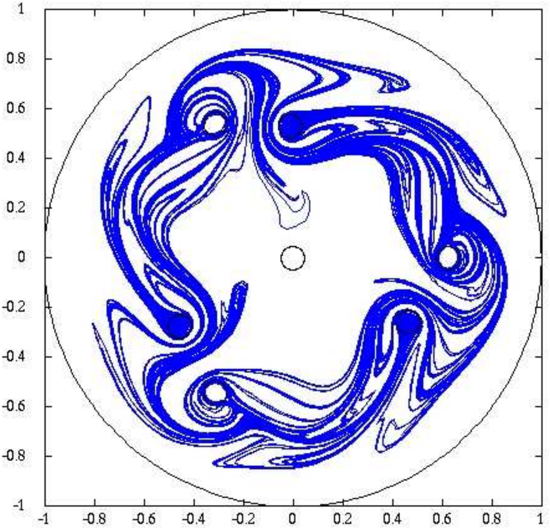
Four Rods



[movie 6] [movie 7]

[M. D. Finn and J.-L. Thiffeault, *SIAM Review*, in press (2011)]

Six Rods



[movie 8]

Some open issues

- The nature of the isotopy between the pA and real system.
- Which orbits dominate? (They live in folds — see for instance Cerbelli & Giona (2006); Thiffeault et al. (2009))
- Sharpness of the entropy bound (progress: linked twist maps — Sturman et al. (2006)).
- Computational methods for isotopy class (random entanglements of trajectories – LCS method).
- 'Designing' for topological chaos.
- Combine with other measures, e.g., mix-norms (Mathew et al., 2005; Lin et al., 2010).
- 3D?! (lots of missing theory)

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